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Original Article

Teacher's conceptualizations of different meanings of pure imaginary numbers *

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Abstract

This study examined in-service teachers' understanding of the pure imaginary number ib, particularly in its Cartesian form. This study was part of a broader design-based research study, which involved the development of a professional development (PD) program aimed at exploring five in-service teachers' understanding of various forms of complex numbers. Data collection included pre- and post-written sessions along with interviews after the PDs. Data pointed to change in teachers' conceptualization of i where some could not reason algebraically or geometrically initially. Upon completion of the PD, however, all participants identified i as one of the roots of the quadratic equation, $x^2 + 1 = 0$ and were able to represent it geometrically as the point (0, 1) on the Complex plane. Additionally, all participants recognized the operator interpretation of i as a 90-degree rotation. One participant also noted dilation meaning of b when multiplied with i and another participant reasoned on the repeated addition meaning. The results further highlighted specific challenges teachers faced in conceptualizing the pure imaginary number. Collectively, the results underscore the importance of addressing the pure imaginary part of the Cartesian form and the operator meanings of complex numbers in teacher education. Furthermore, these results suggest that quantitative reasoning could serve as a foundational way of thinking for making sense of complex numbers, including the unit i.

Keywords

Cartesian form of Complex numbers

Teacher knowledge

Quantitative reasoning

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Introduction

School curricula at different levels require working with different types of number systems as understanding the relationships between these systems is essential (CCSM, 2010). Among these, the complex number system is the most comprehensive, since it extends the real number system. The significance of complex numbers spans mathematics, physics, engineering, and various applied fields, making them a particularly important topic in science, technology, engineering, and mathematics (STEM) education. In particular, complex numbers are crucial in the learning of advanced mathematics and physics topics as well as in the learning of concepts of different fields of engineering including quantum physics (Karam, 2020), relativity, electromagnetic theory, signal processing (Atmaca et al., 2014), hydrodynamics and electrical circuits (Benítez et al., 2013). Therefore, a robust conception of complex numbers has an important role in both accessing to and being successful in interdisciplinary fields (Anevska et al., 2015).

In literature, it is argued that a comprehensive understanding of complex numbers requires interpreting them both geometrically and algebraically. Geometrically, a complex number can be viewed as a point on the Complex plane or as a vector (Fauconnier & Turner, 2002). Algebraically, it is an expression in the form $a + ib$ that "should be conceptualized as one number, i.e., the expression $a + ib$ is a single entity combining a real number and an imaginary number" (Nordlander & Nordlander, 2012, p. 633). According to Sfard (1991), complex numbers can be considered as mathematical objects within a well-defined set where all elements share a common structure. Some researchers point to the challenges students face in conceptualizing complex numbers. Glas (1998) suggests that complex numbers often appear to be formal, abstract constructs to students that lack intuitive meaning or connection to real-world experiences. This abstraction can make it difficult for students to visualize complex numbers or understand their practical relevance. Nordlander and Nordlander (2012) further note that students frequently question the real-life applicability of complex numbers and struggle to grasp what they represent. To address these difficulties, researchers emphasize that students have a need to see the imaginary unit to consider any number as a complex number. Therefore, developing a meaningful comprehension of the Cartesian form, particularly the pure imaginary component, is crucial. In this paper, we specifically focus on teachers' conceptualization of the different meanings of the pure imaginary component within the Cartesian form.

Sfard (1991) argued that conceptualizing complex numbers requires a person firstly to recognize that $i = \sqrt{-1}$, in the Cartesian form. Kontorovich (2018a) further stated that the radical sign √ "…initiates polysemy-a phenomenon in which the same concept or symbol can be interpreted in discrepant manners depending on the context in which they are used and on the curricular norms associated with the context" (p. 17). For instance, in the field of real numbers, √9 is equal to 3, aligning with the definition of a function where each input has a unique output. In contrast, within the complex number system, the square root becomes a multi-valued function and both -3 and 3 are considered to be roots of 9 (Kontorovich, 2018b). In this regard, scholars argue that i must be comprehended as one of the square roots of (-1), requiring a shift in the classroom discourse where the expression '$\sqrt{-1}$ is not a number' no longer holds (Nachlieli & Elbaum-Cohen, 2021). Furthermore, conceptualizing of i as a vector and a point $(0, 1)$ is essential (e.g., Karakok et al., 2015). This geometric interpretation connects also the unit i with the ib component of the Cartesian form, where b is any real number. Particularly, conceptualizing ib as the multiplication of i with a real number can lead to interpretation of i "as a rotation of the real line through 90°" (Harding & Engelbrecht, 2007, p. 967), which produces Complex plane. This perspective further clarifies that real numbers are a subset of complex numbers and points the isomorphism between the Complex plane and the Cartesian plane, which can lead to an understanding as complementary rather than conflicting representations (Kontorovich et al., 2021). Therefore, researchers suggest that it is beneficial for both students and teachers to flexibly shift between these geometric and algebraic interpretations (Kontorovich et al., 2021).

Mathematics teachers are key figures in educating students who need to understand complex numbers, especially in STEM-related fields (NCTM, 2000). In this regard, we investigated teachers' different conceptualizations of i and ib both algebraically and geometrically before and after the completion of a PD study. This study contributes to the literature in the following ways.

First, there are few studies on the conceptualization of pure imaginary numbers. However, for example, a study on teachers' conceptualization of different forms of complex numbers revealed that teachers had difficulty in making sense of pure imaginary numbers as both points and vectors (Karakok et al., 2015). Researchers also reported that teachers merely considered complex numbers as algebraic manipulations done with i. In addition, some eminent researchers have theoretically discussed the construction of the Cartesian form of complex numbers (e.g., Harel, 2013; Sfard, 1991) and examined it from an algebraic point of view (e.g., Nordlander & Nordlander, 2012; Panaoura et al., 2006). Also, in a previous study (Karagöz Akar et al., 2023b) working with the same teachers in the same PD, we reported on only the preliminary results after the completion of the study. In addition, we mostly discussed teachers' dwelling on real number concepts while reasoning with multiplication of i with real numbers, concluding a biased reasoning with complex numbers. Regarding the data on teachers' biased reasoning, in this study, providing a detailed analysis, we elaborated on and explicated further reasons as to how teachers might have the complex number bias.

Secondly, deviating from all the aforementioned research, in this study, we not only reported on data before and after the PD providing a development on the teachers' different conceptualizations of i and ib but also on their difficulties. Here, contributing further to the field, we specifically reported on the changes in teachers' conceptualization of i as one of the roots of $x^2 + 1 = 0$ both algebraically and geometrically before and after the PD, which was regarded as an important step in understanding complex numbers (Sfard, 1991) and was reported as a missing knowledge base of high-school students (Kontorovich, 2018b; Nachlieli & Elbaum-Cohen, 2021). Also, comparing data from different teachers, we reported on the meanings of ib as rotation operator, dilation operator and repeated addition of multiplication.

Third, the meaning that teachers attributed to the units of i and ib both as vectors and points in the complex plane was examined from a quantitative reasoning perspective. Finally, previous studies emphasized that complex numbers historically emerged from mathematicians' consideration of the roots of cubic polynomials (e.g., Harel, 2013; Nahin, 2010). In this study, examining complex numbers as roots of quadratic equations through quantitative reasoning supports the statement that "complex numbers are the only roots that any polynomial equation has!" (Harel, 2013, p. 35) and is consistent with the Fundamental Theorem of Algebra. In the following section, we elaborate on how we extend the field further and how we envision complex numbers through the lenses of quantitative reasoning.

Conceptual Framework

Literature on Complex Numbers

A comprehensive understanding of complex numbers requires knowledge of algebraic and geometric representations (see Table 1) across Cartesian, polar and exponential forms. This understanding also involves recognizing the connections between these representations and having the flexibility to transition among them (Karakok et al., 2015).

Table 1. Representations of Different Forms of Complex Numbers (Reproduced from Karakok et al., 2015, p.329).

Representation	**Form**			
	Purely imaginary	**Cartesian**	**Polar**	**Exponential**
Algebraic	i, $\sqrt{-1}$, $(0,1)$	$a+ib, (a,b), z$	$r(\cos\theta + i\ \sin\theta)$, z	$re^{i\theta}$
Geometric	A point on the complex plane, a rotation operator	A point on the complex plane, a vector with a magnitude of $\sqrt{a^2+b^2}$ and an angle of $\tan^{-1}(\frac{a}{b})$ with the positive real axis, a rotation and dilation operator	A point on the circle centered at the origin with radius r, a vector with magnitude of r and an angle of θ with the positive real axis, a rotation and dilation operator	A vector with magnitude of r and an angle of θ with the positive real axis, a point on the circle centered at the origin with radius r, a rotation and dilation operator

However, existing research highlights the challenges that both students (Çelik & Özdemir, 2011; Nordlander & Nordlander, 2012; Panaoura et al., 2006; Soto-Johnson & Troup, 2014) and mathematics teachers face when working with complex numbers. Specifically, several studies report that teachers often struggle to establish connections between different representations (Conner et al., 2007; Karakok et al., 2015; Nemirovsky et al., 2012). This has led researchers to emphasize the importance of integrating both algebraic and geometric perspectives when teaching and learning complex numbers.

Soto-Johnson and Troup (2014) investigated how mathematics majors reason algebraically and geometrically about complex-valued equations. Their findings revealed that students predominantly relied on algebraic reasoning. However, when explicitly prompted to consider the geometric aspects of the equations, they were able to engage with these perspectives effectively. Therefore, authors concluded that encouraging students to reason both geometrically and algebraically fosters a deeper integration of the two forms of reasoning (Soto-Johnson & Troup, 2014). Similarly, Nordlander and Nordlander (2012) conducted a study involving engineering undergraduates and high school students and found that many students struggled to grasp the fundamental nature of complex numbers—specifically, the idea that any number can be considered a complex number. The researchers argued that making the imaginary unit i explicitly visible is crucial for helping students conceptualize numbers as part of the complex system (Nordlander & Nordlander, 2012). In another study, Nachlieli and Elbaum-Cohen (2021) explored twelfth-grade secondary school students' understanding of complex numbers. They emphasized that a critical aspect of this understanding involves recognizing that "..the word number also signifies objects of the type $a + ib$, where a and b are real numbers, and i is one of the square roots of (-1)..." (p. 5). Their findings suggest that when teachers actively question and prompt students to engage in reflective and investigative thinking, it can facilitate a discursive shift from real to complex numbers, allowing students to reason about these numbers in both algebraic and geometric terms. Further supporting these observations, Panaoura et al. (2006) found that secondary school students tend to view the algebraic and geometric representations of complex numbers as separate entities rather than as alternative forms of the same mathematical object. The researchers suggested that this difficulty may stem from the way complex numbers are typically introduced, with minimal emphasis on visual or geometric interpretations (Panaoura et al., 2006).

Research on secondary mathematics teachers (Conner et al., 2007; Karakok et al., 2015) has also highlighted challenges in their conceptualization of complex numbers, particularly regarding the Cartesian, polar, and exponential forms. In a study with three in-service teachers, Karakok et al. (2015) found that one teacher struggled to visualize complex numbers as points on the Complex plane. For instance, when asked to represent i geometrically, the teacher was uncertain whether it was located one unit above the origin. Another teacher viewed i merely as a symbol and described complex numbers in terms of algebraic manipulations involving i. Both teachers also experienced difficulty in connecting the vector representation to the Cartesian form. These findings suggest that the teachers primarily perceived complex numbers through algebraic operations rather than through their geometric interpretations. Similarly, prospective teachers faced challenges in making connections between complex numbers and their roots of quadratic equations (Conner et al., 2007). In another study, Nemirovsky et al. (2012) investigated prospective secondary mathematics teachers' geometric reasoning about complex numbers. By using a physical classroom floor setting, they encouraged teachers to explore the geometric meaning of addition and multiplication of complex numbers. This hands-on approach enabled the participants to conceptualize multiplication by i as a 90-degree rotation on the Complex plane. Building on these findings, Saraç (2016) worked with a prospective secondary mathematics teacher and examined how she developed the Cartesian form of complex numbers through quantitative reasoning. The results revealed that the participant successfully conceptualized complex numbers as a single entity within a well-defined set—specifically, as the roots of any quadratic equation with real coefficients. Notably, contrary to Karakok et al.'s (2015) findings, this prospective teacher accurately identified i as the point (0,1) on the Complex plane (Saraç & Karagöz Akar, 2017).

In addition to these studies, researchers have suggested several instructional approaches to enhance students' understanding of complex numbers. For example, Edwards et al. (2021) proposed using 3D graphical representations to visualize the complex zeros of a quadratic function. Murray (2018) recommended employing geometric transformations, specifically using reflections from the vertex of a quadratic function, to help learners distinguish between real and non-real (imaginary) roots. Other studies emphasize the value of digital tools such as Computer Algebra Systems (CAS), Computer-Aided Assessment Systems (CAA) (Gaona et al., 2022), and GeoGebra (Caglayan, 2016; Seloane et al., 2023). These technology-based and visual approaches have been shown to improve students' ability to conceptualize complex numbers and to understand the roots of both quadratic and complex equations from a geometric perspective.

The aforementioned studies indicate that teachers and students need to develop a robust understanding of complex numbers paying special attention to the algebraic and geometric meanings of complex numbers, and specifically of the unit i. Although providing important insights, these studies did not specifically focus on the different meanings of pure imaginary numbers. In this study, we report on in-service teachers' conceptualizations of pure imaginary numbers. In addition, we propose that quantitative reasoning might provide a robust thinking process in the development of teachers' conceptions of complex numbers. We do this in the following ways: First, we situate any complex number from the perspective of quantitative reasoning framework (Thompson, 1990, Thompson & Carlson, 2017). With this perspective, in this paper, we consider complex numbers as the quantification of the roots of any quadratic equation with real coefficients (Karagöz Akar, et al., 2024; Karagöz Akar et al., 2023, Karagöz Akar et al., 2023a; Saraç & Karagöz Akar, 2017). In this respect, as recommended in the literature (e.g., Kontorovich, 2018b; Nachlieli & Elbaum-Cohen, 2021), we report on data showing the change in teachers' making sense of i before and after the completion of a PD study. Secondly, situating the meaning of i from the point of view of quantitative reasoning, we elaborate on how vectors could be a mediator for making sense of i as a point. This is especially important as in-service teachers were reported having difficulties in making sense of i both as a point and as a vector (Karakok et al., 2015). In addition, the College Board of Mathematical Sciences (CBMS) emphasized that "complex numbers can fall into the chasm between high school and college, with high school teachers assuming they will be taught in college and college instructors assuming they have been taught in high school" (CBMS, 2012, p. 64). Therefore, reasoning about the pure imaginary numbers from the point of view of quantitative reasoning might provide the field a new lens on top of the conventional way of thinking about them as stated in most of the high school curricula such as "know there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + ib$ with a and b real" (CCSSM, 2010, p. 60). It is in this regard that we scrutinized the following research questions:

1. What are in-service teachers' conceptualizations of i algebraically and geometrically before and after a PD program focusing on quantitative reasoning?
2. What different meanings of ib do teachers have algebraically and geometrically upon completion of a PD program focusing on quantitative reasoning?

In the following section, we explain the constructs of quantitative reasoning and how we conceptualize complex numbers through the lenses of quantitative reasoning.

Quantitative Reasoning

Quantitative reasoning refers to an individual's "analysis of a situation into a quantitative structure" (Thompson, 1990; p. 13). Moore et al. (2009) defined quantitative reasoning further as "…the mental actions of an individual conceiving a situation, constructing quantities of his or her conceived situation, and both developing and reasoning about relationships between these constructed quantities" (p.3). Quantity, quantification, and quantitative structure are paramount notions within quantitative reasoning.

In a person's mind, quantity is a measurable attribute of an object (Thompson, 1990). A person has a mental image of an object and its measurable attributes (qualities) (Thompson, 1994) by explicitly or implicitly conceptualizing an appropriate unit. From this perspective, we understand complex numbers as the union of two quantities as directed distances (Karagöz Akar et al., 2024). This understanding arises from analyzing a mathematical object, such as quadratic functions, and breaking it down into a network of quantities and quantitative relationships, which include the roots and the x-coordinate (abscissa) of the vertex with their distances to each other and from the origin (see Figure 1). The quantitative structure is then viewed as a network of quantities and quantitative relationships, where "quantitative relationship is the conception of three quantities, two of which determines the third by a quantitative operation" (Thompson, 1990, p. 11). Therefore, quantitative operations are mental operations that construct quantities. Quantification refers to the entire process of constructing quantities and quantitative relationships. We further elaborate on how we conceptualize complex numbers connected with quadratic functions through quantitative reasoning as follows.

As Moore et al. (2009) emphasized, conceiving a situation requires envisioning an object and its attributes, such that the mental image "could be an image interpreted from a problem statement or a mathematical object (e.g., a graph)" (Moore et al., 2009, p.3). In this regard, the parabolas located on the coordinate system are the object of thought in this study, with elements such as the roots and the abscissa of the vertex having measurable properties. That is, given any quadratic equation $ax^2 + bx + c = 0$ with real coefficients; two roots of the quadratic equation are of the form $x_{1,2} = \frac{-b \mp \sqrt{b^2-4ac}}{2a}$, where $(b^2 - 4ac)$ is called the discriminant (Δ), $-\frac{b}{2a}$ is the abscissa of the vertex and $(\frac{\sqrt{\Delta}}{2a})$ is the distance from $(-\frac{b}{2a}, 0)$ to the roots on the x-axis (Hedden & Langbauer, 2003). $\frac{\sqrt{\Delta}}{2a}$ is regarded as a quantity whenever the learner evaluates the measure of the distance between the abscissa of the vertex and the roots. Similarly, whenever a person evaluates the distances of the roots to the origin, the roots are also considered a quantity.

In this paper, we contend that such a conception of quadratic roots could promote the notion that complex numbers are the roots of quadratic equations with real coefficients, members of a well-defined set (Sfard, 1991). For this we dwell on two main constructs of quantitative reasoning, namely covariation and multiplicative object (Thompson et al., 2017) in the following way.

First of all, algebraically, "we know that all complex numbers have the form $x + iy$, where x and y are real numbers. Real numbers being all those numbers which are positive, negative, or zero" (Panaoura et al., 2006, p. 683). We also consider algebraic representations such as $x + iy$, (x, y), and z and geometric representations such as a point on the complex plane (Karakok et al., 2015). Then, given any complex number, $z = x + iy$, one can assign meanings to x and y such that x refers to $-\frac{b}{2a}$ algebraically and the abscissa of the vertex of any quadratic function geometrically; and y refers to $\frac{\sqrt{-\Delta}}{2a}$ algebraically and the distances of the roots to $(-\frac{b}{2a}, 0)$ geometrically (see Figure 1). Saldanha and Thompson (1998) defined the notion of covariation as someone's "…holding in mind a sustained image of two quantities' values (magnitudes) simultaneously" (p. 299). Then, the notion of a multiplicative object builds on a person's image of the covariation of quantities where a person conceives a multiplicative object such that she "… tracks either quantity's magnitude with the immediate, explicit, and persistent realization that, at every moment, the other quantity (quantities) also has (have) a magnitude(s)" (Stevens, 2019, p. 42). Thus, a person might decompose the quadratic formula and think of x_1 and x_2 considering $\frac{-b}{2a}$ as the x-coordinate of-the-vertex (as well as the symmetry axis), and its distance to the roots, $\frac{\sqrt{\Delta}}{2a}$ located on the real number line (see figure 1). Considering the x coordinate of the-vertex with its distance to the origin and its distance to the roots is then a quantity in the mind of an individual (see Figure 1). Thus, one might conceive the roots of any quadratic equation as a quantity, having a distance to the origin and a distance to the x coordinate of the vertex (i.e., $\frac{\sqrt{\Delta}}{2a}$). In other words, on can conceive three quantities: 1) the distances of the roots and the x-coordinate of the vertex to the origin and 2) the roots' distances to the x-coordinate of the vertex. Then, quantitative relationships among these quantities entail envisioning how the roots and the x-coordinate of the vertex are positioned on the number line as a point and as distances related to each other.

Thus, from the notion of multiplicative object, one might consider that the ordered pair (x, y) or $(-\frac{b}{2a}, \frac{\sqrt{-\Delta}}{2a})$ is a single entity with two components that combines the magnitudes of two quantities simultaneously (Thompson et al., 2017), where each point is an element of the set of roots of any quadratic equation with real-coefficients. In this regard, we consider complex numbers quantitatively as a measure of the distance of roots (i.e., all the possible roots of quadratic equations) from the origin on the Complex plane.

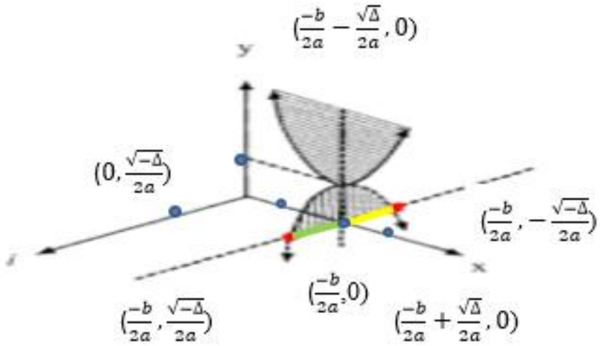


Figure 1. Locating all the roots of a quadratic equation with real coefficients on the plane (Modified from Melliger, 2007)

In this structure, one can also conceive one part of the formula, $\sqrt{-1}\frac{\sqrt{-\Delta}}{2a}$ in two ways: She can envision $\sqrt{-1}$ as one of the roots of $x^2 + 1 = 0$ and assign a numerical value to $\frac{\sqrt{-\Delta}}{2a}$ as 1, considering that the other part of the formula, $\frac{-b}{2a}$, is zero. This might further enable to think of the geometric meaning of $\sqrt{-1}$ as referring to $(0, 1)$ on the imaginary axis with one unit distance above the origin. This image has the potential to trigger an understanding of $\sqrt{-1}$ as a multiplier and $\frac{\sqrt{-\Delta}}{2a}$ as a multiplicand since $\sqrt{-1}$ can be considered as 90 degrees (i.e., a *rotation operator*) counter-clockwise rotation of any real number of the form $(\frac{\sqrt{-\Delta}}{2a}, 0)$ and locating them on the imaginary axis as $(0, \frac{\sqrt{-\Delta}}{2a})$. This image also aligns with what Descartes considered: superimposing the real line onto itself, turning all the points 90 degrees, which would yield the coordinate plane (R^2) (Fauconnier & Turner, 2002). This image can further afford thinking of $\frac{\sqrt{-\Delta}}{2a}$ as a multiplier (the operator) and $\sqrt{-1}$ as a multiplicand as the person knows that $\frac{\sqrt{-\Delta}}{2a}$ is a real number. As stated previously, it is essential to simultaneously consider the algebraic and geometric meanings of complex numbers in order to comprehend any form of complex numbers.

Method

This study is part of a design-based research (DBR) that aimed at enhancing teachers' content knowledge of complex numbers through a PD study. We carried out classroom teaching experiments, followed by a multi-case study (Yin, 2014). In this paper, we present the findings from the multi-case study, emphasizing teachers' existing knowledge base related to their different meanings of the pure imaginary number before and upon completing the PD.

Participants and Data Collection Process

This study involved five secondary school mathematics teachers with 2 to 10 years of teaching experience and degrees in secondary mathematics education. The selection process began with ten teachers completing an open-ended written assessment on complex numbers (pre-written session). In this session, participants were asked to provide algebraic and geometric definitions of quadratic functions and equations, different representations of complex numbers, and vectors. Based on our analysis of their responses, eight participants were purposefully selected according to their background knowledge aligned with the pedagogical goals of our study (Simon, 2000): 1) They demonstrated knowledge of quadratic functions and the definition and expression of vectors. 2) They either did not mention complex numbers in the Cartesian, polar, or Euler forms, or did not explain the relationships between these representations. Nevertheless, five participants declared their availability to attend the study.

As shown in Table 2, the study included four teaching sessions, each lasting between 120 and 150 minutes.

Table 2. Data Collection Procedure

Sessions	Focus of the Sessions
Pre-written session	Analysis of participants' background knowledge
PD session 1	Cartesian form, graphs of and distances within different parabolas
PD session 2	Cartesian form and geometric representation of complex numbers
PD session 3	Definition and properties of vectors and polar form of complex numbers
PD session 4	The Euler form of complex numbers
Post-written session	Analysis of participants' current knowledge
Semi-structured interviews	The participants' current conceptualization of the connections among different forms of complex numbers (30-45 min.)

The first two sessions focused on examining the Cartesian form of complex numbers in relation to quadratic equations through quantitative reasoning (Saraç & Karagöz Akar, 2017). The third session covered the polar form, while the final session addressed the Euler form.

In the context of DBR investigations, the focus was on the theory of quantitative reasoning by examining complex numbers within a quantitative structure. In this structure, complex numbers are understood as a union of two quantities, which are directed distances and are the roots of any quadratic equation with real coefficients. After the participants came to a definition of complex numbers in relation with the roots of any quadratic equation with real coefficients and located them as points on the Complex plane (see figure 1) in PD session 2, specific to the pure imaginary numbers in PD session 3, we asked them to consider the definition of a vector and then think about the qualities of the located points. Once they considered these issues, they could justify how they could think of complex numbers as vectors. At this point, we asked them to evaluate the roots of the equation, $x^2 + 1 = 0$, in terms of the components of the roots of any quadratic equation, namely, $\frac{-b}{2a}$ and $\frac{\sqrt{-\Delta}}{2a}$. They evaluated that numerically $\frac{-b}{2a}$ was equal to 0 and $\frac{\sqrt{-\Delta}}{2a}$ was equal to 1. Given their knowledge of the fact that complex numbers could be represented as vectors with quantities of distances to the origin and with angle measures to the horizontal axis, this allowed them to further conceive that i can be represented as a vector and a point (0, 1) on the Complex plane.

After the PD sessions ended, a post-written session was conducted. Then, the first author, who also facilitated the PD, carried out video-recorded semi-structured interviews to gather data on how participants understood the connections among different forms of complex numbers. The interviews lasted between 30 and 45 minutes. Additionally, participants' written artifacts were collected. Regarding the meaning of i, the pre and the post-written sessions included the following questions "Could you explain what the number i is? Could you show it on the complex plane? Please algebraically and geometrically explain i^2. Justify your answer." Also, in the post-interview, a sample of questions were "How can we represent i geometrically? How do we express i squared (i^2) algebraically? How do we express it geometrically? The multiplication of i and square root minus delta over two a, ($i.\frac{\sqrt{-\Delta}}{2a}$) where is that number on the Complex plane?"

Data Analysis

For data analysis, we employed the constant comparative method (Clement, 2000). The research team collectively reviewed the participants' written responses and transcripts of all interviews, and we watched video recordings when necessary to identify and characterize how teachers conceptualize complex numbers both algebraically and geometrically. Each participant's statements in the transcripts, which ranged from a single sentence to an entire paragraph for each interview question, served as our units of analysis. We conducted the analysis using the premises of quantitative reasoning. Specifically, we focused on how teachers understood complex numbers quantitatively, such as representing a complex number i as one unit distance from the origin, and how they explained its meaning as an operator. We first analyzed the data for each participant individually and then examined the responses to each question across different participants. We created narratives that illustrated how participants conceptualized complex numbers i and ib through quantitative reasoning, comparing and highlighting the similarities and differences in their thought processes.

Results

In the following sections, first we provide results from pre-and post-written sessions related to the different conceptualizations of i which include the definition and geometric representations (See Table 3). Then, we share data from the interview regarding one of the participant's difficulty in locating i on the Complex plane. We also provide data on teachers' responses on the algebraic and geometric meanings of the powers of i before and after the PD. Additionally, we share data on two participants' handling the complex number bias. We finalize this section by presenting different meanings of ib and i with the examples from interview data.

Defining and Representing i as a Point on the Complex Plane

All the teachers could define *i* in the following ways before and after PD as indicated in Table 3.

Table 3. Teachers' Responses Regarding Definition and Geometric Meaning of *i* Before and After PD

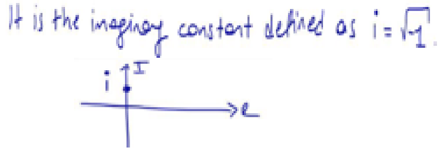
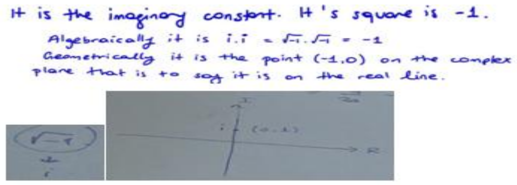
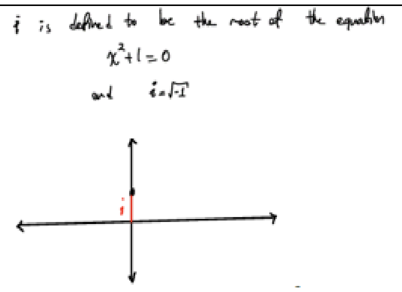
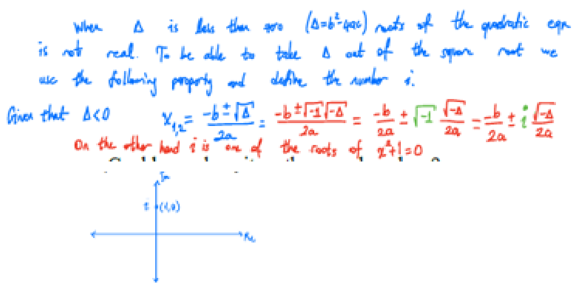
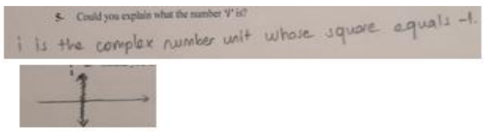
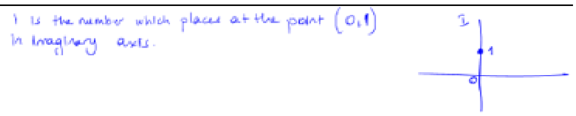
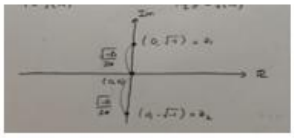
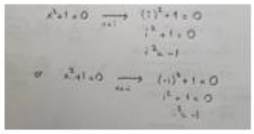
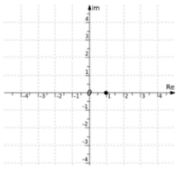
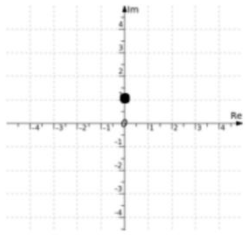
	Before PD	After PD
T1		
T2		
T3		
T4	*i* is an imaginary number which equal to $\sqrt{-1}$. In other words, its square $i^2 = -1$. When the complex numbers are shown in the complex plane x-axis represents the real part and the y-axis represents the complex part. "i" is the square root of -1. In the complex plane that has a one-unit distance from the origin on the y-axis is the place of "i".	"i" and its conjugate "-i" are the roots of the quadratic equation $x^2+1=0$. In the complex plane, these two complex numbers indicate the point that $\sqrt{-1}$ unit distance from the abscissa of the vertex (x=-b/2a) which has the value 0.  

Table 3. Continued

	Before PD	After PD
T5	In real numbers set, square roots of negative numbers cannot be defined because there are no real numbers whose squares are negative. To define them, square root of -1 is defined as imaginary. a. Could you show it on the complex plane? Real part is 0 and the imaginary part is 1. 	The number "i" is one of the roots of quadratic equation $x^2 + 1 = 0$. In real numbers set, square roots of negative numbers cannot be defined because there are no real numbers whose squares are negative. Square root of -1 is defined as imaginary number. a. Could you show it on the complex plane? The number "i" is a complex number where the real part is 0 and the imaginary part is 1. This is how we show it on complex plane: 

As the table indicated, before the PD, all the teachers either defined i as $\sqrt{-1}$ or $i^2 = -1$. Only, T2 pointed to i as the root of the quadratic equation, $x^2 + 1 = 0$. Also, data showed that three of the teachers geometrically pointed to i on the complex plane. In addition, T4 and T5 pointed to the fact that geometrically i is one unit from the origin on the imaginary axis.

On the other hand, after the PD, in their answers in the post-written session, T2, T4 and T5 not only stated that i is one of the roots of the quadratic equation, $x^2 + 1 = 0$ but also could consider $i = \sqrt{-1}$. However, T3, provided the following definition i is the number which places at the point $(0, 1)$ in the imaginary axis". It is important to point that teachers' use of the expression "one of the roots…" together with data on how they located i on the complex plane as (0,1) has merit as teachers need to know that although we accept $\sqrt{-1}$ as the principle root, - $\sqrt{-1}$ is also another root of $x^2 + 1 = 0$. This was specifically shown in T4's written statement such as "i and its conjugate - i".

Similarly, regarding the geometric representation of i, all the teachers except T4 could locate i as a point $(0, 1)$ on the Complex plane. Compared to the data before PD, participants' use of $(0, 1)$ to show i on the complex plane indicated a development on their part. However, T4 wrote $(0, \sqrt{-1})$ and stated that $(0, \sqrt{-1})$ points to $\sqrt{-1}$ unit distance on the imaginary axis. So, we further inquired how she reasoned about placing i on the complex plane.

A Possible Difficulty About Showing i Geometrically

We now present data on T4's reasoning regarding representation of i on the Complex plane during the post-interview. T4 again defined i as one of the roots of the quadratic equation, $x^2 + 1 = 0$. She also stated that i is the number whose square equals -1 and noted that it is equivalent to $\sqrt{-1}$. In her drawing, T4 first represented i by writing it as $(0, \frac{\sqrt{-\Delta}}{2a})$ and then provided an example. She also labeled the points $3i$ and i on the imaginary axis (see Figure 2):

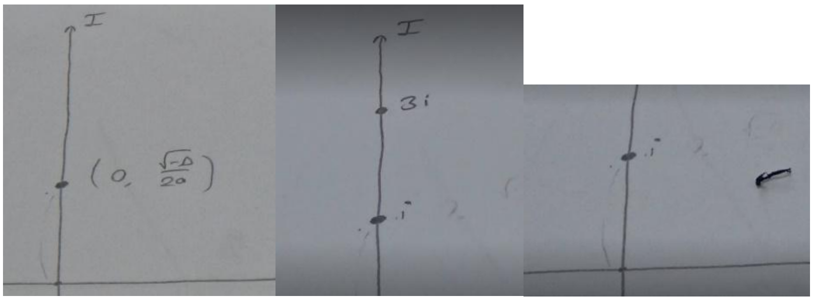


Figure 2. T4's Attempts for Geometric Representation of "i"

Then, hesitantly she questioned if she can use $(0, 3)$ instead of $3i$. She further stated that i was 1 unit above the origin on the imaginary axis.

Discussion continued:

R: Then what happens if you show as a point?

T4: I can't, I don't know, I can't be sure about that $(0, 1)$.

R: Why? You weren't sure about $(0, 3)$ either, were you?

T4: Yes, I wasn't sure about $(0, 3)$ either because we talked about something, for example, we said that we can say $3i$ and i as distance, for example, we couldn't compare them in terms of size. Yes this is 1 unit of distance (referring to the distance between i and the origin) this is 3 units of distance (referring to the distance between $3i$ and the origin in Figure 2) but when I show it as a point, I am not sure if I can write $(0, 3)$ like a coordinate.

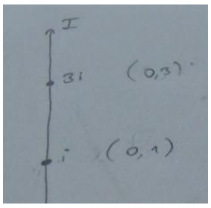


Figure 3. T4 locating i as (0, 1) and $3i$ as (0, 3)

As the data and Figure 2 and Figure 3 indicated, T4 experienced difficulty in representing i as a point on the imaginary axis. She recognized that i was one unit and $3i$ was 3 units above the origin and that they were positioned at specific distances on the imaginary axis. In addition, she understood that in the binomial form, $x+ iy$, the first component referred to the real part, $-\frac{b}{2a}$, while the second component referred to the imaginary part, $(\frac{\sqrt{4ac-b^2}}{2a})$ (see Figure 2). She also noted that in the case of i, the first component was equivalent to 0 and the second component was equal to 1 when considering numerical values. However, from the PD discussions, she remembered that complex numbers cannot be ordered in the same way real numbers can. This realization led her to question whether it was feasible to represent i on the imaginary axis as $(0, 1)$, despite knowing it was one unit above the origin, located on a unit circle. Specifically, she contemplated that if she used $(0, 1)$ or $(0, 3)$ for representing i and $3i$ respectively, it would revert back to real numbers, which can indeed be ordered (for example, 1 is smaller than 3). More importantly, she understood that if she used $(0, 1)$ as shown on $\mathbb{R}^2$, to represent i, it might contradict the understanding that complex numbers cannot be ordered. As a result, she was uncertain if she could express $3i$ as $(0, 3)$. She appeared to be thinking that points on the imaginary axis ought to be represented by imaginary symbols.

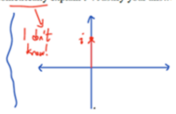
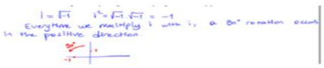
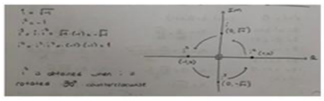
When prompted to reflect on her earlier drawing of the roots on the Complex plane, T4 recalled that complex numbers can be represented as ordered pairs of real numbers. This realization allowed her to represent the binomial form $x+ iy$ as (x, y) on the Complex plane, enabling her to express i as $(0, 1)$. Despite the progress, T4 continued to struggle with comparing the magnitudes of i and $3i$ indicating an incomplete understanding of the magnitude of complex numbers. A significant observation is that T4's attempts to compare real and complex numbers based on the property of ordering are conceptually invalid when viewed within $\mathbb{R}^2$ and the Complex plane. Unlike the real number system, where numbers can be arranged in an ordered sequence, no such inherent ordering exists in $\mathbb{R}^2$ or the Complex plane. This limited understanding suggests that T4 may be applying familiar real-number concepts inappropriately to the more abstract structure of complex numbers.

Meaning of i and b as Operators

Powers of i Algebraically and Geometrically

We share data both from the pre-written and the post-written session in which the participants were asked to explain i^2 both algebraically and geometrically. Data in Table 4 shows that, in the pre-written session, some teachers' answers pointed that although they could write and state i^2 as equal to -1, they did not know how to show it on the Complex plane geometrically. Particularly, for the geometric meaning of i^2, except T5, they stated "I do not know" or they left it blank. In addition, we considered T1's answer as insufficient as we could not tell if she answered the question or if she explained i both algebraically or geometrically. Contrarily, in the post-written session, all the participants could explain the powers of i algebraically by referring to i as equal to $\sqrt{-1}$. All of them also reasoned that i refers to a 90-degree rotation counter clockwise geometrically.

Table 4. Teachers' Responses Regarding Algebraic and Geometric Meaning of i^2 and powers of i Before and After PD

Expressing powers of 'i'	Before PD	After PD
Use of algebraic meaning	T1 $i = 1.i$ that's why we show it as I did in option a. T2 Please algebraically and geometrically explain i^2. Justify your answ $i = \sqrt{-1}$ $i^2 = \sqrt{-1}.\sqrt{-1}$ $= -1$ T3 I do not know. T4 **Left blank.** T5 **Algebraically,** $i^2 = (\sqrt{-1})^2 = -1$	T1 Algebraically it is $i.i = \sqrt{-1}.\sqrt{-1} = -1$ T2 $i.i = \sqrt{-1}.\sqrt{-1}$ $i^2 = -1$ T3 $i = \sqrt{-1}$ $i^2 = \sqrt{-1}.\sqrt{-1} = -1$ T4 $i = \sqrt{-1}$ $i^2 = -1$ T5 **Algebraically,** $i^2 = (\sqrt{-1})^2 = -1$
Use of operator meaning	T1 **Left blank** T2 Please algebraically and geometrically explain i^2. Justify your answ  T3 I do not know. T4 **Left Blank.** T5 **Geometrically,** $i^2 = -1$ **is a number where imaginary part is 0 and real part is -1. On complex plane, the number will be at -1 on real(x) axis.**	T1 Geometrically it is the point (-1,0) on the complex plane that is to say it is on the real line. T2  T3  T4  T5 **Geometrically,** $i^2 = -1$ **is a number where imaginary part is 0 and real part is -1. On complex plane, the number will be at -1 on real(x) axis.**

Complex Number Bias

During the interview, all participants but T1 provided valid explanations as to how they reasoned both algebraically and geometrically. T1 could mention both the operator and the dilation meanings of multiplication of complex numbers. However, her explanations pointed to complex number bias when reasoning about i algebraically. As exemplary, we also provide data from T5 on her reasoning about i algebraically to further compare it with T1's reasoning.

When asked how they interpreted i^2 in the interview, T5 provided the following explanation:

T5: Because i was equal to $\sqrt{-1}$. Therefore, the number we called i^2 became$\sqrt{-1}.\sqrt{-1}$. This gave us the number -1. Minus one is a real number. So, if we write this as a complex number, in the format $x + iy$, it becomes 0+, sorry,$-1 + 0i$. That's why I said we can show it as a number, as a point on the x-axis, at a distance of 1 unit form 0, on the left side, the negative side. So, I said directly it will be on the real axis.

As the excerpt showed T5 could state that i^2 is equal to -1. Her statement about the cartesian form of complex numbers, $x + iy$, and corresponding it with $-1 + 0i$, and thinking about -1 as part of the cartesian form algebraically pointed that she not only realized that real numbers are part of complex numbers but also she knew that as a number -1 referred to 1 unit distance from the origin. This further suggested that she could reason about a power of i quantitatively. However, she did not clarify how she understood or derived the equality $\sqrt{(-1)}.\sqrt{(-1)} = -1$ and the researcher did not ask for further explanation. T5's previous statements, where she identified i as the number whose square equals -1, may have influenced her reasoning for writing this equality.

However, T1's explanations about i^2 pointed to complex number bias:

T1: Let's write (i^2) algebraically. Algebraically, it is multiplying ($\sqrt{-1}.\sqrt{-1}$). Our rules in real numbers are valid here. We can multiply two numbers inside a single root. It becomes ($\sqrt{1}$). So it's one. Um, where do we put it? The result is a real number, so we multiplied two imaginary things, and the answer came out real. Then I'll show on the real (see Figure 4)

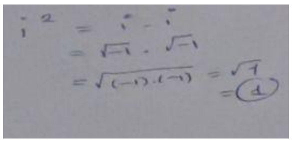


Figure 4. T1 writing about i^2

That's here somewhere (referring to Figure 5), I'm trying to make these things equal. (i^2) will be at $(1, 0)$ isn't it? Yes. Because (i^2) is positive. I mean, I think that we cannot write (i^2) under it (talking about how to label the point $(1, 0)$) because it is (i^2). This is $(1, 0)$. One minute, (i^2) would be -1, right? My mind is burning now. One minute. I multiplied i with i, that is (i^2). The (i^2) has to be -1. Sorry, it won't be like this. This will be -1.

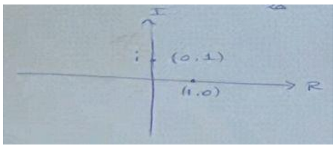


Figure 5. T1 locating i on the Complex plane

As the data showed T1 reasoned with the rules utilized in real numbers such that she considered that $\sqrt{a}.\sqrt{b} = \sqrt{a.b}$ in real numbers also is valid for complex numbers. This suggested that she held complex number bias. This is because, $\sqrt{a}.\sqrt{b} = \sqrt{a.b}$ if a and b are both positive or at least one of them is negative or zero. However, $\sqrt{a}.\sqrt{b} \neq \sqrt{a.b}$ if a and b, both are negative. On the other hand, once she thought of i as the number whose square is -1, she corrected herself. Discussion got more interesting when the researcher asked how T1 reasoned: she corrected herself by giving an example from real numbers, multiplying $\sqrt{2}$ with $\sqrt{2}$ and concluded that she was wrong in saying that $\sqrt{-1}.\sqrt{-1}$ would be equal to 1. She even commented on i^3 and stated that it would be equal to $(0, -1)$ on the Complex plane. Then the researcher asked:

R: Okay. Can you tell me again why you changed your mind here?

T1: I changed my mind here because of this: This is not a real thing (referring to $\sqrt{-1}$). i is not a negative number. i is something like root negative one (see Figure 6)

That's why I changed my mind. Because if i was the number -1, the square of -1 will come out as 1 anyway. There is no problem there. But this is defined as root -1 anyway, there is no such thing normally. But there are two expressions inside the root. Both are the same. It is like, as if the roots have canceled out each other. So it's like it turned into a square. That's why I changed my mind. 'Cause if I do this it'll be like I'm squaring -1. Then it will be as if I have defined it like this.  But it is not like that 

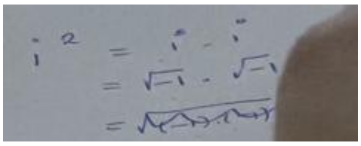


Figure 6. T1's Algebraic Explanation on i^2

Providing an example from real numbers, T1 reasoned that $\sqrt{a}.\sqrt{a} = a$ for any non-negative real number and explained that the result of $\sqrt{(-1)}\sqrt{(-1)} = -1$. However, her explanations still related to the rules held in non-negative real numbers. That is, her explanations were not mathematically valid and she did not provide a legitimate proof. To summarize, T1's explanations indicated that she seemed to think the same properties hold for the radicand in both real and complex numbers.

Different Meanings of "ib"

As we stated earlier, all the participants pointed to multiplying i with i as referring to a 90-degrees counter clockwise rotation after the PD. In this section we provide data from T1 as exemplary for not only to point to how one participant reasoned about the rotation operator but also about the dilation operator meanings of multiplication within complex numbers. It is also necessary to point that as the data below indicated, T1 knew that in the cartesian form of complex numbers, $x + iy$, x referred to $\frac{-b}{2a}$ and y referred to $\frac{\sqrt{-\Delta}}{2a}$. In addition, we provide data from T4 to compare her reasoning to T1's reasoning where T4 seemed to be thinking and transferring the repeated addition understanding of multiplication while reasoning about ib.

Discussion with T1 on the meaning of $\sqrt{-1}.\frac{\sqrt{-\Delta}}{2a}$ followed:

R: This i times $\frac{\sqrt{-\Delta}}{2a}$

T1: Where will it be?

R: What does it correspond to, yeah, where will that number be?

T1: On here (showing a point on imaginary axis). It is on the imaginary axis. It depends on the value of the coefficient. So for instance, if $\frac{\sqrt{-\Delta}}{2a}$ is 3, I'll be pointing at "$3i$". Or like I'll be pointing at $-3i$. It's not a real number (referring to i), but it has a real coefficient. I can say it is getting bigger or smaller according to that. Saying getting bigger or smaller may not be very accurate, as I said, because of the definition of i, it may not be like is i bigger than $3i$, but we can assume its distance as the distance to zero and we can place it (talking about real number times i) Since that is positive, we can place it.

R: Okay. You're telling me… you say that when I multiply it with i, I can place ib here (pointing to the imaginary axis). I'm asking you if you can explain $\frac{\sqrt{-\Delta}}{2a}$ when multiplied with i.

T1 continued her thinking suggesting that she struggled to articulate the geometric meaning of multiplying a real number with i. She even considered that when she multiplied $\frac{\sqrt{-\Delta}}{2a}$ with i, $-\frac{b}{2a}$ was zero and she knew that $\frac{\sqrt{-\Delta}}{2a}i$ was on the imaginary axis. Then, the researcher prompted her twice to explain the meaning of multiplying a real number with i . She then responded:

T1: Okay, for example $\frac{\sqrt{-\Delta}}{2a}$ was on this line (referring to the real axis). Because it was real. When I multiply it with i, shall I say if it rotates, what shall I say? It went this way. It moved to the imaginary axis. It rotated, it rotated. It rotated, from here (meaning the real axis) to here (meaning the imaginary axis). Rotated ninety degrees, yes.

The data revealed that T1 initially interpreted the expression $\sqrt{-1}\frac{\sqrt{-\Delta}}{2a}$ as multiplying i with a real number, focusing on multiplication with a scalar rather than recognizing it as a real number being multiplied with i, which represents a rotation operator. This interpretation may have been influenced by the syntactic structure of the expression, where the order of multiplication symbolically suggests that the real number acts on i. T1 appeared to view $\sqrt{-1}$ as the multiplicand and $\frac{\sqrt{-\Delta}}{2a}$ as the multiplier, implying that the real number scales or dilates i along the imaginary axis. Several factors likely contributed to this understanding. First, during the interview, T1 consistently interpreted the Cartesian form of complex numbers as vectors. She explicitly described i as a unit vector corresponding to (0, 1) and comprehended $\frac{\sqrt{-\Delta}}{2a}$ as a distance from the origin. This vector-based interpretation might have allowed her to position multiples of i as points on the imaginary axis. Second, she referred to $(-\frac{b}{2a})$ as zero and identified $\frac{\sqrt{-\Delta}}{2a}$ as a real number. This suggested that she connected this reasoning to the geometric interpretation of complex roots of quadratic equations, reinforcing her making sense of $\frac{\sqrt{-\Delta}}{2a}$ as a scalar acting on i, dilation meaning of multiplication. On the other hand, the data also indicated that T1 struggled to conceptualize i as a rotation operator. It was only after the researcher explicitly directed her attention to this distinction multiple times that T1 acknowledged the geometric meaning of i as a multiplier. Once prompted, she recognized that multiplying a real number with i rotates it 90 degrees, thereby locating it on the imaginary axis. This shift in understanding suggests that T1 was capable of reasoning about the multiplication of i with real numbers in two distinct ways—both as a dilation and a rotation operator—although she required external prompting to articulate the latter perspective. More importantly, T1's consideration of 3 in $3i$ as a distance to zero and $\frac{\sqrt{-\Delta}}{2a}$ in $\frac{\sqrt{-\Delta}}{2a}i$ as a distance on the real number line suggested that she reasoned on iy component of the cartesian form of complex numbers, $x + iy$ quantitatively. We argue that such consideration yielded T1 to explain her reasoning behind multiplication of i with a real number both as a dilation and rotation operator as she was able to locate both i and y on the complex plane as $(0, 1)$ and $(0, y)$ respectively.

Comparatively, T4 reasoned as the following about $\frac{\sqrt{-\Delta}}{2a}i$ while thinking about the polar form:

T4: Okay, I will say 1 to this (see Figure 7). I add the vector 1 as much as $|z|cos\theta$ and I make the x component from here, when I express it in binomial form it becomes $|z|cos\theta$.. Then I can apply it here, for example, i, (adding) as much as yi. When I combine the units there, $|z|sin\theta$ there, this time I get yi. When I move that y by $|z|cos\theta$ and add it to the end, for example, I arrive at this point (showing the location of the complex number) and there, I can show it as $|z|sin\theta$ too (showing dotted lines in the figure).

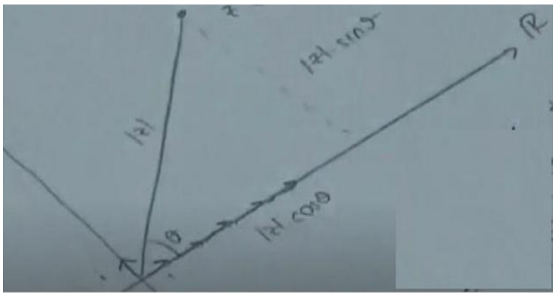


Figure 7. T4 representing repeated addition of $\frac{\sqrt{-\Delta}}{2a}i$ with reference to $i|z|sin\theta$

Data pointed to some important conclusions: First, similar to T1's reasoning, T4's thinking of i as 1 unit on the imaginary axis suggested that she seemed to be thinking iy component of the cartesian form of complex numbers, $x + iy$ quantitatively. Particularly, T4 seemed to consider that $\frac{\sqrt{-\Delta}}{2a}i$ where y is equal to $|z|sin\theta$ is a distance on the imaginary axis. More importantly, comparing to T1' s reasoning, T4's thinking of $\frac{\sqrt{-\Delta}}{2a}i$ pointed that T4 seemed to be repeatedly adding the (1, 0) vector on the real axis to form the x-component while adding repeatedly the (0, 1) vector on the imaginary axis to form the y component of $x + iy$. That is, T1's data showed that her reasoning involved multiplication of a real number with i as a dilation operator whereas T4's reasoning involved multiplication with repeated addition. As a final remark, data also suggested that T4's reasoning on the x component as a vector with repeated addition of the unit vector (1, 0) and on the y component as a vector with repeated addition of the unit vector (0, 1) seemed to enable her to think of $x + iy$ as a vector addition.

Discussion and Suggestions

The study's findings revealed that after completing a PD that focused on the connections between various forms of complex numbers, in-service teachers developed both algebraic and geometric interpretations of i. Specifically, results showed that prior to the PD, none of the teachers could state i as one of the roots of $x^2 + 1 = 0$ whereas upon completion of the PD, all the participants not only could recognize i as the principal root but also could conceptualize it as a vector and a point on the Complex plane. Also, all teachers were able to describe i as a rotation operator. Additionally, one participant demonstrated a more advanced understanding by interpreting ib in two ways: as i performing a 90-degree rotation on b and as b acting on i as a dilation operator. On the other hand, another participant could explain ib by repeated addition of the unit vector (0, 1). Results further pointed to a teacher's biased thinking on the algebraic meaning of $\sqrt{-1}\sqrt{-1}$. In addition, different from earlier research, data showed how in-service teachers could or could not overcome their difficulties on their own considering complex numbers as the quantification of the roots of quadratic equations. We acknowledge that this study was conducted with only five in-service teachers. However, results point to some important insights about how they conceptualized different meanings of i and ib. In what follows, while discussing the results of the study, we also point to the implications for further research on and teaching of the algebraic and geometric meanings of i.

Defining and showing i as a point on the Complex plane

Results showed that T2, T4 and T5 identified i as one of the roots $x^2 + 1 = 0$ in their definitions. As exemplified, T4 even algebraically pointed $\pm i$ as the two roots. In addition, T1 also had an awareness of i as one of the roots of $x^2 + 1 = 0$. Results importantly extend previous research as researchers emphasized that i needs to be conceptualized by students and teachers as one of the square roots of -1 (Kontorovich, 2018b; Nachlieli & Elbaum-Cohen, 2021) since both teachers and students need to be aware of the different interpretations of the radical sign in different number sets (Kontorovich, 2018b). However, results also showed that all participants described $\sqrt{-1}$ as equal to i. Since i is considered as the principal root of $x^2 + 1 = 0$ in formal mathematics, these conceptualizations of the teachers can be considered as valid (e.g., Kontorovich, 2018b; Usiskin et al., 2003).

Going beyond the algebraic manipulations, data particularly pointed that in-service teachers were able to link the roots of $x^2 + 1 = 0$ to their conceptualization of complex numbers as the quantification of the roots of any quadratic equation with real coefficients. That is, as specifically shown by data from T4 in table 3 and in Figure 2 and Figure 7 and in T1's explanations, while considering i and ib they both referred back to the numerical values of $(-\frac{b}{2a}, \frac{\sqrt{-\Delta}}{2a})$ where they stated that $-\frac{b}{2a}$ would be equal to zero. This suggested that they were able to conceive the quadratic formula quantitatively (Stevens, 2019) where they considered such as $\frac{\sqrt{-\Delta}}{2a}$ as the vertical distance from 0. This further suggested that they were able to both consider $(-\frac{b}{2a}, \frac{\sqrt{-\Delta}}{2a})$ as a union of $(-\frac{b}{2a}, 0)$ and $(0, \frac{\sqrt{-\Delta}}{2a})$ but also i as a vector with pointwise representation $(0, 1)$. This suggested that they conceived i as a multiplicative object (Thompson et al., 2014), the union of two quantities. This was specifically evident in T4's overcoming her difficulty in positioning i on the Complex plane. In fact, whenever she thought that as points such as $(0, 1)$ and $(0, 3)$ referred to one of the roots of a quadratic equation, she recalled that she could use ordered pairs of real numbers to show i as $(0, 1)$. Though, the use of $(0, \sqrt{-1})$ in locating one root of $x^2 + 1 = 0$ might be a natural tendency on the part of learners. Taking this into consideration, while teaching quadratic equations, we argue for the simultaneous utilization of both algebraic and geometric meanings of the roots. Specifically, we recommend both teachers and teacher educators to provide opportunities for their students to consider the pointwise representation of the roots, as ordered pairs, on the horizontal axis in $\mathbb{R}^2$.

These findings suggest that understanding $\mathbb{R}^2$ as a quantitative structure may support a clearer comprehension of the isomorphism between $\mathbb{R}^2$ and the Complex plane. This is particularly significant when we consider T4's difficulty, which indicates she may not have recognized that each ordered pair in $\mathbb{R}^2$corresponds uniquely to a complex number, reflecting a one-to-one relationship between these two structures (Kontorovich et al., 2021). Given this, we advocate for creating opportunities for teachers and students to first conceptualize $\mathbb{R}^2$ in a quantitative manner (Karagöz Akar, Zembat et al., 2022), treating points not just as static locations but as multiplicative objects (Saldanha & Thompson, 1998; Stevens & Moore, 2017; Thompson et al., 2017). This approach could enhance their ability to perceive the structural equivalence between the Cartesian plane and the Complex plane.

Results also showed that T1 considered that the rule $\sqrt{a}.\sqrt{b} = \sqrt{a.b}$ in real numbers also is valid for complex numbers, indicating a complex number bias (Karagöz Akar et al., 2023b; Kontorovich, 2018a). In addition, although T5 could verify $\sqrt{-1}\sqrt{-1}$ as equal to -1, there was not enough data to conclude whether she could provide a valid explanation on that. On the other hand, even though T1 could provide an explanation to resolve her difficulty about $\sqrt{-1}\sqrt{-1}$, her explanation further evidenced her complex number bias. Therefore, we comply with Nachlieli and Elbaum-Cohen (2021)'s suggestion and with an emphasis on the polysemy of the radical sign (Kontorovich, 2018a), and further recommend teacher educators and teachers to include a discussion about the importance of taking into consideration of i as one of the square roots of -1, the principal root, while teaching and doing research on different forms of complex numbers.

Meaning of i and b as rotation and dilation operators

In the case of multiplication with i, the results align with and expand on previous research (Nemirovsky et al., 2012; Soto-Johnson & Troup, 2014), showing that all participating teachers could interpret i as a 90-degree rotation operator. This understanding enabled them to visualize the powers of i on the complex plane. Notably, one teacher (T1) demonstrated a more advanced conceptualization of ib by recognizing both the rotation and dilation meanings. Specifically, she could view i as the multiplier and b as the multiplicand, and vice versa. Comparatively, T4's reasoning suggested that she used repeated addition meaning of multiplication while interpreting ib. T1s dual interpretation of ib suggests a deeper understanding of the connection between the geometric and algebraic meanings of complex numbers while the repeated addition suggests a limited meaning, which we call for further investigation.

Previous studies have indicated how physical environments, such as a classroom floor, can support pre-service teachers in understanding the multiplication of a complex number by i, leading them to interpret i as a 90-degree rotation operator (Nemirovsky et al., 2012). However, Soto-Johnson and Troup (2014) found that while two undergraduate mathematics majors recognized that multiplying two complex numbers involves both rotation and dilation, their understanding remained incomplete. Although they could verbally explain the geometric meaning of this multiplication, they struggled to produce corresponding diagrams that connected the geometric interpretation with algebraic representations. Similarly, T1 had difficulty in recalling the knowledge that i, as a multiplier, could act on any real number b by rotating it 90-degree counterclockwise. In addition, Tekin (2019) studied how a pre-service teacher developed a meaning for multiplication of complex numbers by focusing on both rotation and dilation separately and in-juxtaposition to each other. However, the participant in her study had already an understanding of multiplication of i with a real number both as a rotation and dilation operator. Further research is needed to investigate how learners develop an understanding of multiplying i by a real number from both the multiplier and multiplicand perspectives. Clinical design-based research studies, such as teaching experiments, could provide deeper insights into how this dual meaning emerges.

Considering operator meanings of rotation and dilation in multiplication is crucial, as research on rational number multiplication identifies two reasoning models: repeated addition and multiplicative reasoning. The repeated addition model is viewed as limited and elementary (Fischbein et al., 1985), while multiplicative reasoning involves understanding multiplication as "times as much" (Thompson & Saldanha, 2003). This advanced reasoning requires recognizing the product in relation to both the multiplier and the multiplicand (Karagöz Akar, Watanabe et al., 2022). In addition, previous studies indicate that pre-service teachers often struggle to grasp multiplication multiplicatively (e.g., Karyağdı, 2022). The findings from this study, particularly T1's ability to interpret i with both the multiplier and multiplicand meanings and T4's repeated addition meaning, suggest the need to extend research on multiplication to the domain of complex numbers. Investigating how learners conceptualize the multiplication of i with real numbers could reveal new reasoning models and clarify how such understanding develops. This line of inquiry is particularly important, as the teachers' explanations imply that a dual interpretation of multiplication with i may be rooted in a quantitative understanding of quadratic roots, their relationship with vectors, and a quantitative conceptualization of the Complex plane.

In relation with the aforesaid discussion, results also point that at least two teachers, T1 and T4, conceived ib as a single entity quantitatively. This was evidenced in how T1 showed ib as points on the Complex plane by pointing to the algebraic form with regards to the quadratic roots and also in how she reasoned about ib by utilizing multiplication. Similarly, this was shown by T4's reasoning about ib as repeated addition of the unit vector, i. So, we argue that a conception of ib as a single entity quantitatively has also importance from an algebraic point of view. Algebraic structure sense includes seeing an algebraic expression as an entity and dividing an entity into sub-structures (Hoch & Dreyfus, 2004) as well as "seeing the elements of a set as objects upon which the operations act" (Novotná et al., 2006, p. 249). So, a teacher's conceiving ib as a single entity quantitatively suggests that quantitative reasoning might help study secondary teachers' mathematical meanings (Thompson, 2016) for the algebraic structures (Smith III & Thompson, 2007; Thompson 2011). Thus, using the lenses of quantitative reasoning, we further propose research where many domains of secondary teachers' algebraic knowledge are understudied and underspecified (Warren et al., 2016). This is further important since having a coherent picture of how mathematical ideas (Ball et al., 2008) and mathematical structures in the curriculum are connected is essential for teachers (Warren et al., 2016).

As a final note, since this study solely focused on five in-service teachers' conceptualizations of different meanings of i and ib, based on the aforementioned discussion, we propose to do further research on students' and (preservice) mathematics teachers' conceptualizations of different meanings of i and ib with developing hypothetical learning trajectories in design based studies and with a larger number of participants.

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