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Original Article

Investigating prospective mathematics teachers' use of concrete materials in place value concept in different bases: addition and subtraction with whole numbers

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Abstract

The aim of this study is to examine how prospective mathematics teachers (PMTs) conceptualize the place value concept in different number bases and how they utilize concrete materials in this process. To achieve this aim, a case study design was utilized. The participants of this study consist of 24 PMTs from a public university in Turkey. The participants of this study were asked to answer activity questions that required them to perform addition and subtraction operations on numbers written in base ten, base six and base three using at least two concrete materials. Participants completed this activity as a group, with four weeks to provide written responses and the freedom to use any type of concrete material. The findings revealed that PMTs employed not only proportional and non-proportional models, as stated in related literature, but also a mixed model approach. The use of the mixed model emerged as an effective strategy, allowing PMTs to leverage the strengths of both proportional and non-proportional models. Another finding indicated that PMTs were limited in generating solutions using a second concrete material. This limitation highlights the difficulties PMTs face in maintaining material diversity when working with different base systems, which in turn affects their ability to construct mathematical meaning.

Keywords

Place value
Concrete material
Addition and subtraction operations
Base arithmetic
Middle school mathematics prospective teachers

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Introduction

Children do not come to school knowing nothing about the concept of numbers; in fact, they already have many ideas about it. Natural numbers and operations are fundamental topics in the mathematics curriculum (Ministry of National Education of Turkey (MoNE), 2018; National Council of Teachers of Mathematics (NTCM), 2000). Place value, which is included in these topics, is an important concept that helps students learn arithmetic and algebra (Conference Board of The Mathematical Sciences, 2001). Before the concept of place value, children learn numbers and make sense of them by recognizing patterns when counting sometimes between 10 and 20 (Van de Walle et al., 2018). When representing numbers, perceiving 10 objects as a single entity and representing them in groups according to their place values help make sense of place value. As students advance through their

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schooling, the focus of instruction transitions from counting individual units to counting by grouping items into sets of 10, a process referred to as unitizing. Unitizing sets of 10 marks the formal introduction of place value concepts (Rojo et al., 2021). In a numeral, the size of the group is determined by the position of the digit within the numeral, for instance in 729, the '7' signifies seven hundreds, while in 174, the '7' represents seven tens (Findell et al., 2001). Because place value refers to the idea that a digit's value is determined by its position within a number (Reys et al., 2014).

Basic understanding of the values represented by digits is also important for comprehending different number systems and algorithms (e.g., addition and subtraction) performed in different number systems (Fasteen et al., 2015; Thanheiser, 2009). A rich understanding of the number system is only possible with a solid grasp of the multiplicative structure required for understanding operations on numbers (Nataraj & Thomas, 2009). It is crucial to understand "the marriage of place value" (Hose & Wells, 2013, p. 528) in counting.

Understanding place value concept through the base 10 system begins from the moment students start learning mathematics (MoNE, 2018) and students encounter at the high school or university level under the topic of base arithmetic (Council of Higher Education (CoHE), 2007). As children continue to work on patterns of numbers between 10 and 20, counting up to 100 and discovering patterns between numbers, the construction of a sense of place value also continues. Physical models, grouping numbers, understanding the place value of digits, and comprehending the relationships between them, play a key role in the process of understanding the base 10 system (Rojo et al., 2021; Van de Walle et al., 2018). Games that children play with concrete materials (counting chips, number charts, base ten blocks etc.) support the idea of 10 and beyond becoming widespread (Van de Walle et al., 2018). Van de Walle et al.'s (2018) proportional and non-proportional models for place value represent two distinct approaches to understanding numerical place value. Proportional models visually and physically represent numerical magnitudes in direct proportion to their values; for example, base ten blocks concretely illustrate the magnitude each block represents. In contrast, non-proportional models depict numerical values without a direct correlation to physical size; for instance, tokens or colored rods may represent numbers, but their physical dimensions do not correspond to the numerical value. These two models enable students to grasp the concept of place value in different ways, fostering a deeper understanding of the concept.

Since students work on the base 10 number system in the elementary school curriculum, prospective teachers are also prepared to teach using the base 10 number system. Prospective teachers can have the opportunity to work on other number systems in the specialized area knowledge courses they take (CoHE, 2018). Although the place value is an important part of the elementary school curriculum, it is also one of the topics that prospective teachers have difficulty in conceptual applications (Roy, 2014). Research results show that prospective teachers successfully apply procedures that use the place value, but they fall short in understanding the procedures they apply (Tarım & Artut, 2013). The prior knowledge or readiness of prospective teachers has revealed that they can correctly state the places of numbers, but they cannot transfer this idea to inter-place relationships and operations (McClain, 2009). Therefore, talking about the 10-times relationship between places while working on the base 10 may not actually show that prospective teachers understand this base and the concept of place value conceptually. For example, the results of Thanheiser and Rhoads' (2009) study revealed that only about 30% of prospective teachers came to class with prior knowledge of interpreting the numbers 1 obtained as a carry-over in the tens and hundreds place correctly. According to Thanheiser's (2005) study, only 20% of prospective teachers can interpret the values of digits and the relationships between digits in the base ten system. The researcher believes that the borrowing approach that stands out in exchanging between digits masks their weak understanding of the base ten system. According to him, understanding the grouping represented individually by digits (for example, 300 is 3 groups of 100; 60 is 6 groups of 10, and 7 is 7 groups of 1) is as important as reading the number correctly (367 - three hundred sixty-seven). However, prospective teachers cannot make this grouping idea or effectively use this relationship.

Research studies have also revealed that prospective teachers who work with different bases face some difficulties in the concept of place value. Thanheiser and Rhoads' (2009) study, which worked on a different basis, addressed the processes of understanding the number system of the Mayans, who work in the base 20 number system, and relating it to the base ten system for prospective teachers. One of the results of the study revealed that more than half of the prospective teachers had difficulty understanding the grouping method used by the Mayans working in the base 20 and could not correctly write their three-digit numbers. They interpreted the zero number as filling a gap rather than evaluating it according to the position it is located in. In this process, it has also been revealed that when prospective teachers work with numbers in the base 10 system, the underlying structure of numbers written in base-10 is not deeply understood and remains implicit. Similarly, Yackel et al. (2007), who worked on a different base than the base ten system, implemented a series of mathematical tasks, including the base 8, for primary school prospective teachers in their first field course for developing a deep conceptual understanding of early arithmetic for six weeks. The results of the study showed that the language supporting the place value in the base 8 could be learned through tasks that allow for the application of this language. This process, which started with noticing basic number facts in the base 8 with visual material support, was developed through tasks emphasizing the relation between the digits. In McClain's (2009) study, which aimed to demonstrate the multiplicative relation between place values and the transformations made in addition and subtraction operations, prospective teachers seemed to be reasoning in connecting with the base ten system. However, it was revealed that they could not transfer their base ten thinking to base eight number representations correctly. On the other hand, with a series of sequential mathematical tasks, Roy (2004), who worked in the base 8, asked prospective teachers to perform addition and subtraction operations in the octal system with contextual problems and the support of an empty number line. In this process, it was determined that prospective teachers performed different counting strategies on the number line, made exchanges among the digits of the numbers given to them in the octal system, and they used conceptual problem-solving strategies commonly. This study revealed that prospective teachers made less operational and more in-depth conceptual relations and explanations because of experiencing tasks that support the conceptual value of digits. Prospective teachers were surprised to discover that the algorithm for addition and subtraction operations was the same in different number bases. Fasteen et al. (2015) worked with prospective teachers using the base 5 system to help them explore the underlying structure of different number systems through multiplication. In that study, a lesson was designed to help prospective teachers understand multiplication in the base 5 system using concrete materials. This approach enabled them to grasp the difference between the representation of the number 10 in base 10 and base 5. Understanding 10 in base 5 as a single quantity rather than two digits aided in comprehending multiplication using the area model. Additionally, this method helped prospective teachers develop a conceptual understanding of the positional structure and multiplicative relationships within the base 5 system.

This related literature clearly highlights the difficulties prospective teachers face in understanding the concept of place value in different base systems or what has been done to improve their understanding. While these studies focus on how prospective teachers develop conceptual thinking strategies when working with different bases, they do not deeply investigate how prospective teachers comprehend place value through the use of concrete materials. However, we know that concrete materials provide an opportunity to make the multiplicative relationship between place values tangible, thus offering a discussable context for this relationship (Cady et al., 2008; Rusiman & Him, 2017; Thompson & Lambdin, 1994). Because one of the ways of multiple representation is to learn or teach mathematics through concrete materials. These representations can be symbolic, visual, verbal, or physical, and they play a critical role in the teaching and learning of mathematics. According to Lesh et al. (1987), multiple representations are essential for deep understanding because they allow students to see mathematical ideas from different perspectives and make connections between concepts. For instance, a concept like place value can be represented using base-ten blocks, number lines, or algebraic notation, each offering a unique way to interpret the concept. Goldin and Kaput (1996) also highlight the significance of multiple representations, suggesting that they support students' ability to communicate mathematical ideas more effectively. By using diverse forms of representation, students can develop a richer and more flexible understanding of mathematical concepts, which is crucial for both learning and teaching mathematics (Duval, 2006).

Ball and Bass (2000) emphasize the importance of intertwining content and pedagogy in teacher education. Prospective teachers need to engage with the same challenges their future students will face, particularly in understanding place value (Yackel et al., 2007). One effective approach to achieving this is by having prospective teachers solve arithmetic tasks in unfamiliar bases, which fosters empathy for their students' struggles and deepens their understanding of place value concepts (Roy, 2014). This understanding is facilitated through the use of concrete materials, as adults also benefit from tangible aids when learning new concepts (Mix, 2010). By incorporating concrete materials as part of multiple representations, this process makes the understanding of place value more accessible for prospective teachers, grounding abstract concepts in concrete experiences. Accordingly, the aim of this study is to examine how prospective mathematics teachers (PMTs) conceptualize the place value concept in different number bases and how they utilize concrete materials in this process. Based on this, the research question of this study is: How do PMTs use concrete materials to conceptualize the place value concept in different number bases, and what strategies do they adopt in this process?

Method

In this study, a case study design, typical of qualitative research, was used (Yin, 2003). In this design, an in-depth investigation of a specific case is conducted. This process requires gathering sufficient data on a specific individual, social environment, occurrence, or collective in order for the researcher to comprehend the mechanics or influence behind it (Berg, 2001, p.225). In the context of this study, PMTs were taken as the case of the research.

Participants and Context of the Study

The participants of this study consist of 24 junior prospective mathematics teachers (4 male, 20 female) who took one of the teaching mathematics courses naming Special Teaching Methods 2, which delve into the fundamentals of elementary and middle school mathematics instruction. It covers teaching methods, materials, strategies, and the latest research in teaching mathematical concepts to students, modern teaching approaches, recent curriculum updates. This course further requires prospective middle school mathematics teachers to create lesson plans and evaluations on a range of subjects. Moreover, in that course, they are guided to learn early number concepts and number sense, meanings for the operations, basic facts, whole-number and place-value concepts, strategies for addition and subtraction, multiplication and division computation, and algebraic thinking. Prior to this course, participants had completed courses in General Mathematics, Abstract Mathematics, Geometry, Linear Algebra I-II, Analysis I-II, Mathematical Misconceptions, and Special Teaching Methods 1 (CoHE, 2007). Notably, these university-level courses did not include specific content related to base arithmetic. However, as base arithmetic is included in the student selection and placement examination, teacher candidates typically enter university with prior exposure to this topic.

The sample of the study was determined within the scope of purposive and accessible sampling (Fraenkel et al., 2012). The criteria for study participation were enrollment in the Special Teaching Methods-2 course and prior experience in using concrete materials in middle school mathematics. In addition, students who took this course were accessible participants for the researchers.

These students were given the opportunity to freely use the materials in the math laboratory within an activity related to the development of the concept of numbers in the early stages, and then they had the experience of using base ten blocks within the scope of the place value topic. In a project aimed at revealing participants' mathematical knowledge and awareness through the use of concrete materials, a set of activity questions, partially presented in Table 1, was provided. Participants were asked to respond to these questions in writing, and reflective thinking reports were used to understand their experiences during this process. PMTs were given four weeks to answer the activity questions and were free to use any type of concrete material. During this process, they were allowed to use the math laboratory where the course was held in their free time. This laboratory was established as a classroom for using concrete materials individually or in groups, as recommended by the Turkish Ministry of National Education for middle school level. The following materials were currently available in the math laboratory: a student drawing set (compass-protractor-ruler) (50 pieces), ten frames set (15 pieces), geometric shapes (3 boxes), fraction set (round) (5 boxes), pattern blocks (plastic) 256 pieces (2 boxes), unit cubes (2 boxes), connecting unit cubes (2 boxes), geometric strips (2 boxes), base ten blocks (3 boxes), symmetry mirrors (10 pieces), transparent fraction cards (5 boxes), tangram (2 boxes), algebra tiles set (2 boxes), transparent counting chips (3 boxes), and wooden sticks (1000 pieces). If they wanted to use any material that was not available in this classroom, it was stated in classroom announcements that this was possible.

Participants were allowed to complete this activity as a group. The study was conducted with PMTs who voluntarily chose this assignment from among various topics they completed as part of a project. In this scope, a total of 6 groups of PMTs (24 PMTs in total) produced solutions for the activity. In this study, the PMTs who were already divided into groups were assigned group numbers, represented by the letter "G" followed by a number (e.g., G1, G2, ..., G6).

Data collection tool

This study is a part of a research project which consists of four questions and their sub-questions prepared for the operations of addition, subtraction, multiplication and division. In this study, an activity aimed at enabling PMTs to explore the concept of place value through different bases provided to them, was used as a data collection tool which was represented in Table 1. Departing from the relevant literature (e.g: Ev Çimen, 2016; Fasteen et al., 2015; Roy, 2014; Yackel et al., 2007), the PMTs were asked to demonstrate their approach to addition and subtraction operations with numbers given to them in different bases, base 10, base 3, and base 6, through concrete materials.

The questions within this activity served two separate purposes. Firstly, it aimed to enable PMTs to understand the conceptual dimension of the different bases they will encounter, as well as the addition and subtraction operations that will allow them to make this discovery. PMTs will need to understand the exchange operations they perform in numbers written in the base-ten system. to make sense of these operations. In this context, by including numbers written in different bases that they are not familiar with, but that include the same numbers they are familiar with in base 10, PMTs were given the opportunity to make this discovery using their prior knowledge. In addition, since place value models can be modeled in two ways in the literature (proportional models and non-proportional models), PMTs were asked to use at least two different concrete materials. The questions presented to PMTs are shown below:

Table 1. Questions about operations in base 10, base 6 and base 3 presented to PMTs

Each of the following numbers is given in the base 10. If you wanted to show these operations using at least two different concrete materials, how would you do it? (Don't forget to take pictures of your solutions and explain them in detail.)	
$\begin{array}{r} 10202 \\ -\ 1121 \\ \hline \end{array}$	$\begin{array}{r} 4302 \\ +\ \ 505 \\ \hline \end{array}$
The following numbers are given in different number systems. If you wanted to show these operations using at least two different concrete materials, how would you do it? (Don't forget to take pictures of your solutions and explain them in detail.)	
$\begin{array}{r} (10202)_3 \\ -\ (1121)_3 \\ \hline \end{array}$	$\begin{array}{r} (4302)_6 \\ +\ \ (505)_6 \\ \hline \end{array}$

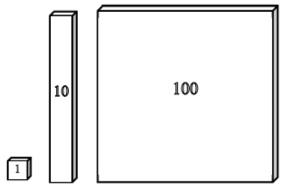
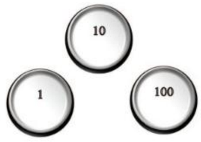
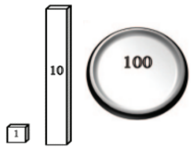
As seen in Table 1, in the measurement tool, two questions for each of addition and subtraction operations were asked. According to Findell et al. (2001), understanding the concept of place value through operations enables the comprehension of the value carried by numbers based on their positions, a concept that becomes particularly evident in arithmetic operations. These operations require students to actively engage with place value by utilizing the positional significance of numbers. For instance, when the value in the hundreds place of one number is added to the value in the hundreds place of another number, the student must focus on the placement order and the value carried by each digit. This process transforms the concept of place value from an abstract idea into a concrete application. Such operations require students to practically test their numerical knowledge in understanding place value. Addition and subtraction provide a less complex and more fundamental foundation for deepening the understanding of place value compared to multiplication and division. Therefore, in this study, the examination of place value is presented through addition and subtraction operations. One of the questions were asked in base 10 and the other was in the same number but represented in a different base. Just as elementary and middle school students may not be familiar with the base 10 number system when developing their understanding of place value, the use of different bases in this activity was a new situation for PMTs.

Data analysis

Content analysis was used to examine the solutions of the PMTs. In the content analysis process, first, the solutions presented by the PMTs were evaluated holistically, and their similarities and differences were noted. Then, parallel to the purpose of the study, they were coded within the scope of "Models for Place Values" mentioned by Van de Walle et al. (2018). "Models for Place Values" approach by Van de Walle et al. (2018) is a well-established widely recognized in the field of mathematics education. This model is grounded in a rich theoretical foundation and has been employed by various researchers (Disney & Eisenreich, 2018; Houdement & Petitfour, 2019; Rojo et al., 2021) to explore students' understanding of place value concepts. Moreover, Hiebert and Carpenter (1992) emphasize the importance of considering the distance between materials and mathematical representation when using concrete manipulatives in the process of understanding place value. According to them, an approach in which grouping is explicitly evident—referred to by Van de Walle et al. (2018) as proportional modeling—allows students to physically experience the relationships between place values. In contrast, an approach that does not provide direct physical cues—termed non-proportional modeling by Van de Walle et al. (2018)—requires students to establish this relationship in a more abstract manner. In this regard, proportional models offer learners greater contextual support, whereas non-proportional models function as more abstract representations (Baroody, 1990). Given the alignment between our research objectives and the strengths of this model, we believe it provides a robust framework for analyzing our data.

Accordingly, if PMTs perform the representations of numbers and the trades, they make during the operations proportionally using place values, it is coded as proportional model usage, and if they perform it non-proportionally, it is coded as non-proportional model usage. Some PMTs, on the other hand, have made sense of place values while performing the representations and operations of numbers without relying on a single model. Data in this form was coded as mixed model usage. An example of each of these models is presented below:

Table 2. Coding definition for the models of representing place value concept through concrete models*

Models for representing place value	Coding definition*	Pictorial representations
Proportional model	Strategies that have number representations and operation descriptions in which the physical relationship between place values is clearly seen.	 (Rojo et al., 2021, p. 37)
Non-proportional model	Strategies in which the physical relationship between place values does not exist in number representations but exists in operation descriptions.	 (Rojo et al., 2021, p. 37)
Mixed model	Strategies in which the physical relationship between place values exists in some digits of number representations, but not in others.	 (the current data of the study produced an additional use getting benefit from Rojo et al. (2021 and Van de Walle et al. (2018))

* Developed from Van de Walle et al. (2018, p.191-193)

The validity and reliability of the study

The content validity was provided by the support of the second author and the related literature. The first researcher initially created the questions, and then the second researcher reviewed them. The questions were developed and refined through input from both authors and their collaborative discussions throughout the process. In this process, the selection of numbers was employed to prompt PMTs to comprehend the multiplicative connections within place value and to devise methods for representing transactions (Ev Çimen, 2016; McClain, 2009; Roy, 2014). Additionally, to contribute to the bases 8, 20 and 5 studied in the literature, the base selection was made differently from these. To gain a more profound insight into the PMTs' conceptual understanding, the researchers decided to demonstrate the concept using at least two different materials, aligning with the guidance in the literature (Van de Walle et al., 2018). After developing the data collection tool, the first author collected the data in the form of a project assignment.

Reliability of the study was ensured through intercoder agreement (Saldaña, 2012). Firstly, two groups were randomly selected, and their activities were analyzed simultaneously by the authors. In this way, coding list was created. Following this, the remaining four activities were independently coded, and the researchers conducted four separate sessions for these four activities. In each session, areas that were coded differently or proposed changes to the coding list were evaluated, and updates to the data analysis were determined and implemented for the next analysis and for the relevant areas in the previous analyses.

Ethics

The procedures performed in studies involving human participants were in accordance with the comparable ethical standards and approved by the Applied Ethics Research Center of the university where the study was conducted.

Findings

Distribution of PMTs' use of concrete materials regarding operations, base numbers and the variety of them

Table 3 shows that G1 and G5 were able to use two types of concrete materials for each of the bases while G2, G3, and G4 were able to use one material for each base. We can say that there may be problems in increasing the variety of materials regardless of the base. Table 3 also revealed that G6 was able to use one material for subtraction in base 10 but was unable to use any material for addition in base 6. Therefore, we can say that increasing the variety of concrete materials can become problematic when the base changes.

The table also clearly shows which materials PMTs used. PMTs used algebra tiles (A.T), wooden sticks (W.S), unit cubes (U.C), connecting cubes (C.C), geometry strips (G.S), or base ten blocks (B.T.B) to represent the operations. Accordingly, it shows that algebra tiles, unit cubes-connected unit cubes, wooden sticks, and geometry strips were the least preferred concrete materials by the groups for representing place value. In contrast, unit cubes, connected unit cubes, and base ten blocks were the most frequently chosen materials. Among these materials, base ten blocks were the only material used across all different bases. Only one group (G4) used unit cubes and connected cubes together in their representations.

Table 3. The distribution of variety of materials* regarding operations and base numbers

Operations	**Addition**		**Subtraction**	
Groups**	**Base 6**	**Base 10**	**Base 3**	**Base 10**
G1	A.T, B.T.B (2 materials)	A.T, B.T.B (2 materials)	U.C, C.C (2 materials)	G.S, B.T.B (2 materials)
G2	B.T.B (1 material)	B.T.B (1 material)	B.T.B (1 material)	B.T.B (1 material)
G3	C.C (1 material)	B.T.B (1 material)	C.C (1 material)	B.T.B (1 material)
G4	U.C – C.C (1 material)	B.T.B (1 material)	U.C – C.C (1 material)	B.T.B (1 material)
G5	U.C, C.C (2 materials)	W.S, B.T.B (2 materials)	U.C, C.C (2 materials)	W.S, B.T.B (2 materials)
G6	N/A***	G.S, B.T.B (2 materials)	U.C, C.C (2 materials)	B.T.B (1 material)

*A.T:Algebra tiles, W.S: wooden sticks, U.C: unit cubes, C.C: connecting cubes, G.S: geometry strips, B.T.B: base ten blocks, U.C – C.C: unit cubes and connecting cubes (which were used together); ** G1-G6: Different groups of participants; ***N/A: Not Applicable which address that group did not offer any representation

Table 4, shows how the prospective teachers modeled place value using different materials. Accordingly, the findings of the study revealed that the participants employed three different modeling strategies to represent place value while using the concrete materials: proportional modeling strategy, non-proportional modeling strategy, and mixed modeling strategy.

Table 4. The distribution of representation approaches regarding the variety of the materials* and base numbers

Representation approaches	**Addition**		**Subtraction**	
	Base 10	**Base 6**	**Base 10**	**Base 3**
Proportional modeling	B.T.B	B.T.B	B.T.B	B.T.B, U.C, C.C
Non-proportional modeling	W.S, G.S	U.C, C.C	W.S, G.S	U.C, C.C
Mixed modeling	A.T	A.T, B.T.B, U.C-CC	-	U.C-CC

**A.T:Algebra tiles, W.S: wooden sticks, U.C: unit cubes, C.C: connecting cubes, G.S: geometry strips, B.T.B: base ten blocks U.C – C.C: unit cubes and connecting cubes (which were used together)

According to this table, proportional modeling involved the use of base-ten blocks, unit cubes, or connecting cubes. It can be stated that base-ten blocks were used consistently across different bases in this modeling approach. In non-proportional modeling, unit cubes, connecting cubes, wooden sticks, or geometry strips were used. Non-proportional modeling was the only representation where the base-ten blocks material was not utilized. In mixed modeling, algebra tiles, base-ten blocks, or a combination of unit cubes and connecting cubes were employed. Algebra tiles were exclusively used in mixed modeling, whereas unit cubes and connecting cubes were applied across all types of modeling strategies. For operations in the base ten, regardless of the specific operation, base-ten blocks, wooden sticks, and geometry strips were preferred. Unit cubes and connected unit cubes were the concrete materials most frequently used in base 3 and base 6.

PMTs' place value representations for the operation of 4302 + 505 in bases 6 and 10

4302 + 505 was presented to PMTs to be solved with at least two concrete materials in both base 10 and base 6. In the following, we closely examine the usage of proportional, non-proportional, and mixed model for this operation.

Representing Approaches in the Base 10

Utilizing proportional model in the base 10. As seen from Table 4, the use of proportional model for addition operation in the base 10 has only been represented through base ten blocks. An example was represented in Figure 1 below. G4 emphasized in their explanations that they started to make addition from the units place and moving to the left. Accordingly, 5 ones plus 2 ones equals 7 ones. As there are no tens in either number, the sum has 0 in the tens place. Following this, they obtained 8 hundreds in the hundreds place and 4 thousand in the thousands place. Therefore, the sum is 4807, consisting of 4 thousand, 8 hundreds, 0 tens, and 7 ones.

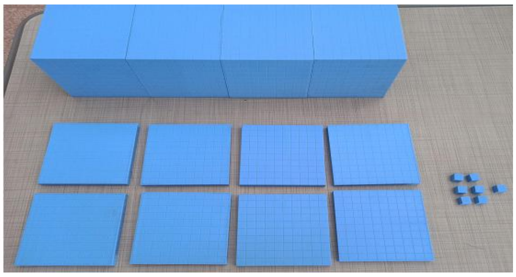


Figure 1. Proportional modeling through base ten blocks for 4302 + 505 (G4)

Utilizing non-proportional model in the base 10. PMTs used non-proportional modeling for base 10 addition through wooden sticks (G5) and geometry strips (G6). According to this, G5 has assigned the numerical values of thousands, hundreds, and ones to identical wooden sticks, respectively, as shown in Figure 2; while G6 has assigned the numerical values of thousands, hundreds, and ones to different colored geometry strips, as seen in Figure 3.

Figure 2. Non-proportional modeling through wooden sticks for 4302 + 505 (G5)

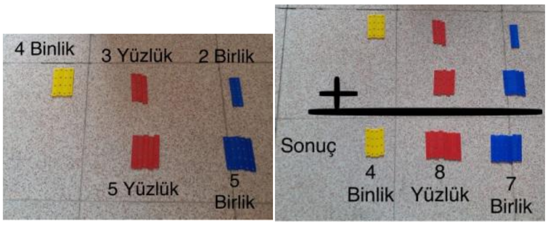


Figure 3. Non-proportional modeling through geometry stripes for 4302 + 505 (G6)

This method of representing numerical values of digits with materials provided an abstract representation rather than directly concretizing the relationships between place values of digits.

Utilizing mix model in the base 10. G1 used the algebra tiles through a mixed modeling approach to perform the given addition operation. As shown in Figure 4, in this usage, the hundreds and ones were modeled with a non-proportional modeling approach, while the thousands were modeled with a proportional modeling approach with the equivalence representing ten hundreds.

Accordingly, the group first added 2 and 5 in the ones place to obtain 7 units and represented this with 7 small square algebra tiles. Since both numbers had 0 in the tens place, they obtained 0 as the result. In the hundreds place, when they added 3 hundreds and 5 hundreds, they represented 8 hundreds with 8 large square algebra tiles. Finally, in the thousands place, they modeled 4 thousands as four stacks, each consisting of ten hundreds. In other words, the group used ten green tiles representing a hundred for each of the 4 thousands in the number 4302. Through this process, they obtained the number consisting of 7 thousands, 8 hundreds, 0 tens, and 4 units.

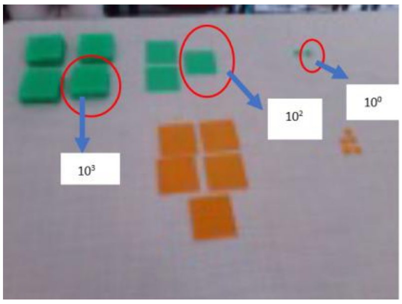


Figure 4. Mix modeling approach through algebra tiles for 4302 + 505 (G1)

The PMTs took advantage of the ability to represent the thousands place, the highest place value, more easily and perhaps to represent the relationship between the hundreds and thousands places more clearly through proportional representation. We can also say that they utilized the flexibility of representing relatively smaller numbers in the ones and hundreds places with fewer materials through non-proportional modeling. The manipulative they used is not suitable for establishing a physical multiplicative relationship between the ones and tens places or between the tens and hundreds places. However, G1 found it appropriate to show this multiplicative relationship only between the hundreds and thousands places.

Representing Approaches in the Base 6

Utilizing proportional model in the base 6. The use of proportional model for addition operation in the base 6 has only used by G2. The group represented how the addition was performed using base ten blocks in Figure 5. We understand that they grouped the blocks for the digit 6^3 as 216, for the digit 6^2 as 36, and for the digit 6^1 as 6 on pieces of base ten blocks. For instance, for representing 216, they used two hundreds block, one ten block and six-unit blocks. In the end, as seen in Figure 5, they represented the result of the operation through the material.

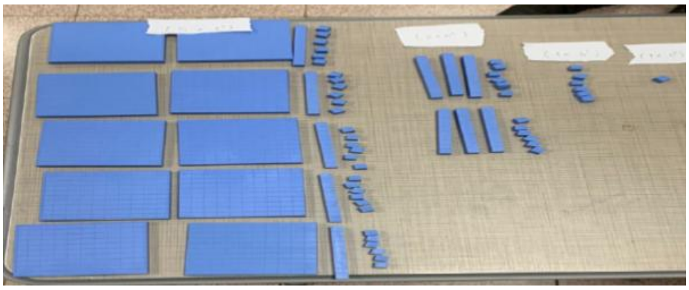
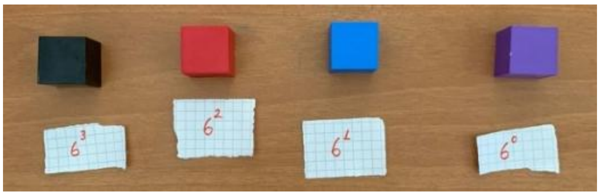


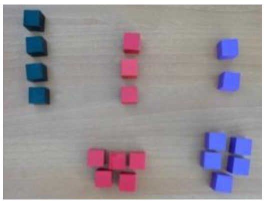
Figure 5. Proportional modeling through base ten blocks for $(4302)_6 + (505)_6$ (G2)

The proportional modeling approaches used in base 10 and base 6 share a foundational logic, where numbers are added according to their place values, with carry-over to the next place value as necessary. In the base 10 model, PMTs typically employed base ten blocks to represent the numbers and directly summed the values at each place without requiring any carry-over. This made the process straightforward and less complex. In contrast, the base 6 model also utilized base ten blocks, but with the added complexity that when the sum at any place value reached 6, a carry-over to the next place value was necessary. For example, when adding 6 and 1, a remainder of 1 was recorded, and 1 was carried over to the next place value. This made the base 6 operations more intricate and required careful attention to the concept of carrying over.

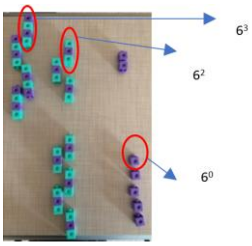
Utilizing non-proportional model in the base 6. In the addition process in base 6, G5 preferred to use non-proportional model. Although the materials used were different, the common idea in this model was the use of materials with non-proportional correspondences to represent the place values of the given numbers in base 6. The group assigned each digit of the number to a different color of unit cubes or connected unit cubes (Figure 6a and c). Then, the calculations were performed using unit cubes and connected unit cubes to represent the numbers without carrying out the exchanges (Figure 6b and d). In their explanations, the students focused on the use of the non-proportional model and detailed the steps of the operation. Accordingly, when adding 5 units and 2 units, they determined that they obtained a total of 7 units, at which point they converted 6 units into a single unit of the next higher place value, and carried over 1 unit to the next column. Similarly, when they reached 8 in the 36s place, they converted 6 groups of 6^2 into one unit of 6^3, and transferred it to the 216s place, ultimately arriving at the final result of 5211. Throughout this process, they utilized the six-times relationship between place values to perform exchanges. While the PMTs' ability to clearly articulate the steps of their operations suggests that they could make sense of their procedural processes, it also raises the concern that they might be limited in demonstrating the multiplicative relationships and exchanges between place values through the use of manipulatives.



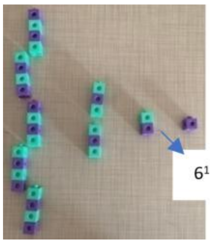
a. Representing digits of the base 6 with unit cubes



b. Representing the numbers $(4302)_6 + (505)_6$ with unit cubes



c. Representing the numbers $(4302)_6 + (505)_6$ with connected unit cubes

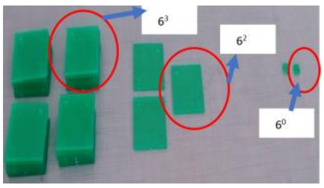


d. Representing the result of $(5211)_6$

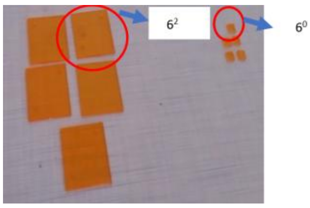
Figure 6. Non-proportional modeling through unit cubes and connected unit cubes for $(4302)_6 + (505)_6$ (G5)

When examining the representations made with unit cubes in Figure 6a, we observe that unit cubes of different colors, despite being the same size, are used to represent different place values. However, when these different colors are used to represent each addend (Figure 6b), the corresponding place values do not match with the unit cubes of the appropriate color. Accordingly, Accordingly, each addend in the 6^0 place is represented by a blue cube, which corresponds to the 6^1 place. In the other representation using connecting unit cubes (Figure 6c and d), both different colors and varying numbers of connecting unit cubes were used to represent the addends and the total place values. The increase in the number of connecting unit cubes as the place value increases (1 purple, 1 purple and 1 green, 2 green and 1 purple, 2 green and 2 purple) seems to have been used as a mental cue by the PMTs. We can suggest that these representations support the visualization of the numerical values of the places directly, but the visualization of the place values themselves in a less direct manner.

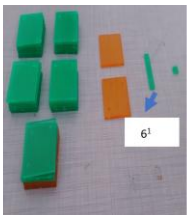
Utilizing mix model in the base 6. G1 has used a mixed modeling approach for addition in the base 6 system. They arranged the algebra tiles in base-6 as the square algebra tile represents 6 × 6 units or the 6^2 place, the rectangular algebra tile represents 6 × 1 units or the 6^1 place, and the smallest algebra tile represents the 6^0 place or a single unit. As can be seen from these explanations, although a proportional perspective was initially adopted, the multiplicative relationship was reflected in the manipulative only between the 6^2 and 6^3 place values. As shown in Figure 7, the group modeled 6^3 digits, with a height of 6. Since the green x^2 algebra tile represents 6^2, the 6^{3rd} place digit in this model is also proportional to a height of 6. However, the relationships between the other place values were assumed to be proportional but were represented non-proportionally. The green tile is associated with the 6^2nd digit and the small green tiles are associated with 6^1st digit using a non-proportional approach.



a. Representing $(4302)_6$ through algebra tiles



b. Representing $(505)_6$ through algebra tiles

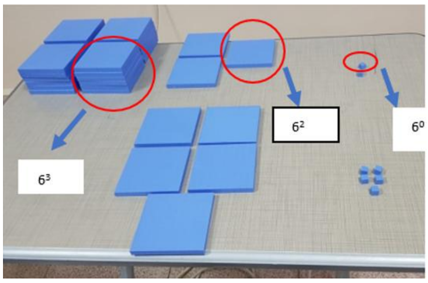


c. Representing the result of $(5211)_6$ through algebra tiles

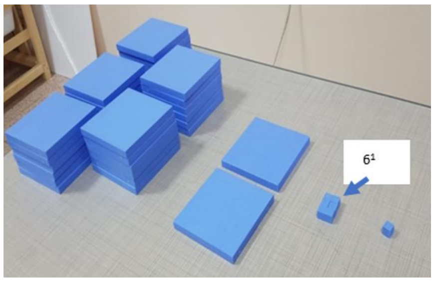
Figure 7. Mix modeling approach through algebra tiles for $(4302)_6 + (505)_6$ (G1)

G1 used a mixed approach in their use of second material choice as well. It would be more accurate to say that they used a proportional approach in a mixed way on the material. The same group employed a similar approach with base-ten blocks using a mixed model, as they did with the previous use of algebra tiles. They arranged the base-ten blocks by modeling their sizes after 6^0, 6^2, and 6^3, corresponding to the ones, hundreds, and thousands places, respectively.

Figures 8a shows the representations of numbers in base 6 using base ten blocks. The representation of 2 in the 6^0th place and the representation of the 6^1 place value with 6 unit cubes are proportional. Instead of the base equivalent of 36 for the 6^2nd place, one hundred block was used as if it is base 10 representation. Therefore, each 6^2 corresponds to one hundred, just as the hundreds place corresponds to 10^2. Figure 8b shows the representation of the number obtained after the exchange operations between 6^1, 6^2 and 6^3.



a. Representing $(4302)_6$ and $(505)_6$ with base ten blocks



b. Representing the result of $(5211)_6$ through base ten blocks

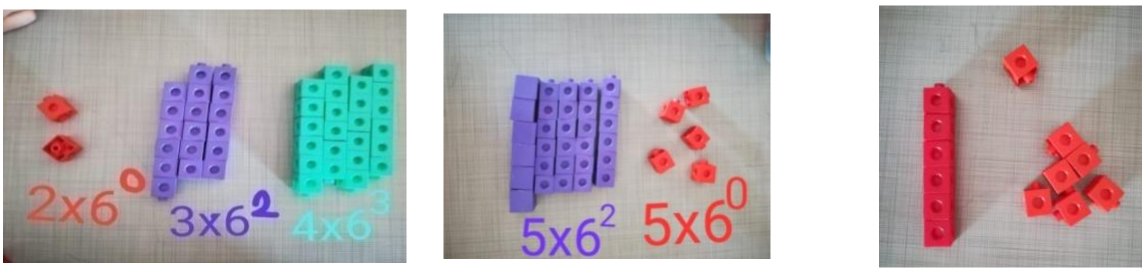
Figure 8. Mix modeling through base ten blocks for $(4302)_6 + (505)_6$ (G1)

G3 is another group who utilized mixed approach. The group used a combination of unit cubes and connected unit cubes. The number of cubes appears to be assigned to the base number. The digit representations of G3 are represented in Figure 9. Accordingly, 6^0 and 6^1 share a proportional relationship in both color and quantity whereas 6^1, 6^2 and 6^3 address non-proportional relations among each other. Therefore, G3 utilized mixed approach for $(4302)_6 + (505)_6$.

6^0 6^1 6^2 6^3

Figure 9. The digit representations for $(4302)_6 + (505)_6$ (G3)

As represented in Figure 10, it was observed in Figure 10-a that the PMTs wrote the digit placement of the numbers differently. The group wrote the digits of the other addend $(505)_6$ in their correct places (Figure 10-b). Although the equivalence between 6^0 and 6^1 was explained using the materials (Figure 10-c), the transitions between 6^0 and 6^1, 6^2, and 6^3 were not represented by them.

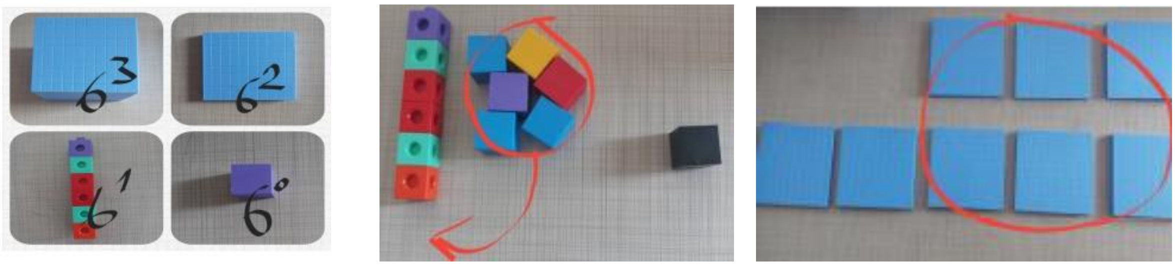


a. Representing $(4302)_6$ with connected cubes	b. Representing $(505)_6$ with connected cubes	c. Representing the exchange between 6^0, $6.\ 6^0$ *and* 6^1

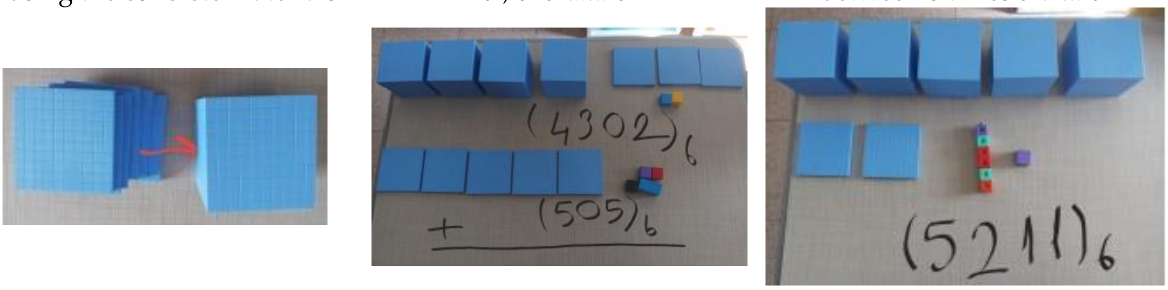
Figure 10. Mix modeling through unit cubes and connected cubes for $(4302)_6 + (505)_6$ (G3)

The absence of a visual material for the inter-place value changes, except for the 6^0 place in G3, suggests that they performed the operation through procedural transformation and determined the result accordingly.

The other group, G4 utilized mixed approach. The group used base ten blocks, connected cubes, and unit cubes to represent ones and 6^1s places proportionally (Figure 11-a, b), but the other places (Figure 11a-c-d) using a non-proportional modeling approach. As shown in Figure 11, they used base ten blocks to represent the 6^2 and 6^3 places as numerically unequal but pseudo-equivalent:



a. Representing the digits of base 6 using the concrete materials
b. Representing the exchange of 6^0, $6.\ 6^0$ *and* 6^1
c. Emphasizing the exchange between 6 times 6^2 *and* 6^3



d. Emphasizing the equality of 6 times 6^2 *means* 6^3
e. Representing $(4302)_6$ and $(505)_6$ with the concrete materials
f. Representing the result of $(5211)_6$ with the concrete materials

Figure 11. Mix modeling through unit cubes, connected cubes, and base ten blocks for $(4302)_6 + (505)_6$ (G4)

G4 emphasized the exchanges between the digits in a base-6 numeral system through concrete materials. Additionally, the group modeled the numbers $(4302)_6$ and $(505)_6$ using concrete materials and demonstrated the result of their addition as $(5211)_6$. This method enabled students to visualize the different place values through concrete materials.

PMTs' place value representations for the operation of 10202 – 1121 in Bases 3 and 10

Representing Approaches in the Base 10

Utilizing proportional model in the base 10. Subtraction operation at the base 10 has been performed by proportional model approach by all groups through the base ten blocks. G1 describes how subtraction is performed by decomposing numbers using base-10 blocks. Initially, the number 10202 was decomposed into ten thousands, hundreds, and units. The number to be subtracted, 1121, was separated into thousands, hundreds, tens, and units. The subtraction process began with the units place; subtracting one unit from two units left one unit remaining. Since there were zero tens, two tens could not be subtracted, and thus, one hundred was converted into ten tens to continue the operation. These steps were similarly repeated for other place values, resulting in the final answer of 9081 as represented in Figure 12.

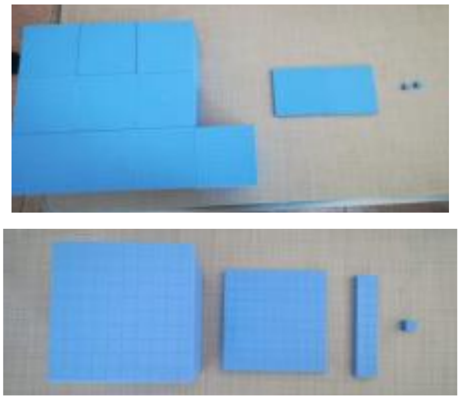


Figure 12. Proportional modeling through base ten blocks for 10202-1121 (G3)

Utilizing non-proportional model in the base 10. PMTs have provided non-proportional modeling for subtraction in base ten using coffee sticks (G5) and geometry strips (G1) as represented in Figure 13.

a. Representing 10202 and1121 with wooden sticks

b. Representing the result of 9081 with wooden sticks

c. Representing 10202, 1121, and the result of 9081 with geometry strips

Figure 13. Non-proportional modeling through wooden sticks (G5) and geometry strips for 10202-1121 (G1)

As shown in Figure 13a and b, G5 associated baskets with place values and sticks with the numerical values of the digits. Specifically, the pink basket was associated with the unit, hundreds, and thousands digits, while the green basket referred to the tens and thousands digits.

Another material used in this representation is geometry strips. G1 used geometry strips to represent the longest strip as thousands, medium-length strips as tens, and the shortest strips as units, as shown in Figure 13c. Unlike the use by G5, the geometry strips representing the values of the digits have different colors, and no additional material was used to represent the place values.

Representing Approaches in the Base 3

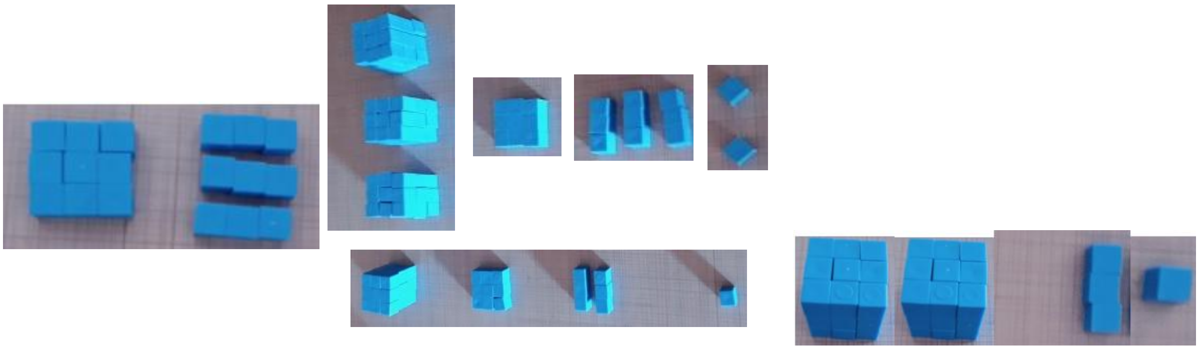
Utilizing proportional model in the base 3. The use of proportional model for subtraction operation in the base 3 has applied by G1, G2, and G6. The groups represented the operation using base ten blocks, unit cubes and connected unit cubes. All the ways of utilizing the concrete materials were shown in Figure 14, 15, 16, and 17.

G1 has performed the solution by creating the place values in base 3 using unit cubes as shown in Figure 14 and 15. According to this approach, the group combined three unit blocks from the base-ten blocks to represent 3^0 as a whole and used it to denote 3^1. They represented 3^2 by arranging nine unit blocks into a 3 × 3 square. For 3^3, they formed a cube, while for 3^4, they placed three separate 3^3 cubes (Figures 14a and 14b). The number $(1121)_3$ results in a model represented by one unit, two groups of three, a square of size (3 × 3), and a cube of size (3 × 3 × 3) (Figure 14a).

a. Modeling $(1121)_3$ b. Modeling $(10202)_3$

Figure 14. Proportional modeling through unit cubes for $(10202)_3$ and $(1121)_3$ (G1)

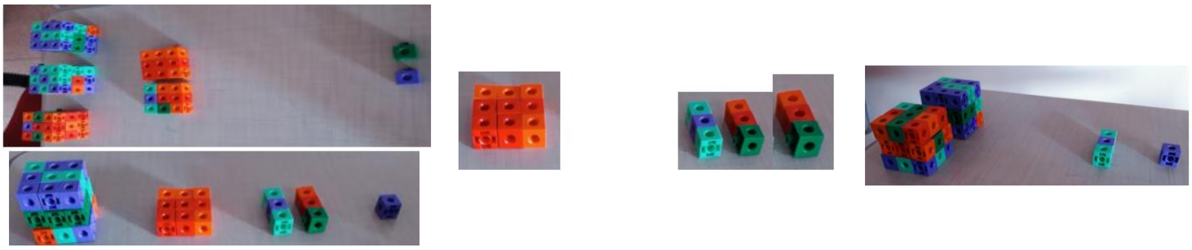
The subtraction of the numbers was modeled by exchanging larger base cubes for smaller units when necessary, reflecting the process of borrowing in base 3. Accordingly, participants initially performed a multiplicative pattern between the 3^2 place and the 3^1 place, as demonstrated in Figure 15a. Subsequently, a multiplicative pattern between the 3^4 place and the 3^3 place was performed, as shown in Figure 15b. Finally, by utilizing this multiplicative pattern, the result was visualized as shown in Figure 15c.



a. Modeling the multiplicative pattern between 3^2 and 3^1 b. Modeling the multiplicative pattern between 3^4 and 3^3 and representing $(10202)_3$ and $(1121)_3$ c. Modeling the result of 10202-1121 as $(2011)_3$

Figure 15. Proportional modeling through unit cubes for the exchanges and the result of $(10202)_3$ - $(1121)_3$ (G1)

As shown in Figure 16, G1 has performed the same steps in connected unit cube as well. Indeed, we can understand that G1 did not consider two kinds of approaches, a non-proportional modeling or mix model approach, rather the group has flexibly used the proportional approach in two kind of materials. Therefore, it can be said that it was not always the case that PMTs developed an alternative approach for using a second material.



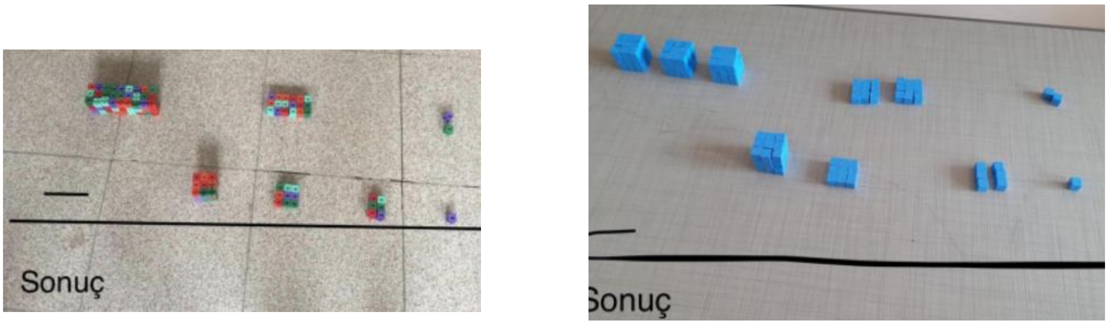
a. Modeling $(10202)_3$ and $(1121)_3$

b. Modeling the multiplicative pattern between 3^2 and 3^1

c. Modeling the result of $(10202)_{3}$- $(1121)_3$

Figure 16. Proportional modeling through connected unit cubes for $(10202)_{3}$- $(1121)_3$ (G1)

The modeling of G6 is identical to G1's modeling except for one point. The only difference is that in the representation of the number 102023, they show the 3^4 digit as 1.3^4 instead of 3.3^3, as shown in Figure 17a even though in Figure 17b, it is represented as 3.3^3:



a. Representing $10202)_3$ and $(1121)_3$ with connected cubes

b. Representing $10202)_3$ and $(1121)_3$ with unit cubes

Figure 17. Proportional modeling through connected unit cubes and unit cubes for $(10202)_3$- $(1121)_3$ (G6)

The G2 group has only represented the solution proportionally in the presentation of the material (Figure 18). The result of the operation, which is 20113, has been modeled as 2.3^3, 1.3^1, and 1.3^0 based on 2 groups of 27, 1 group of 3, and 1 unit of 1.

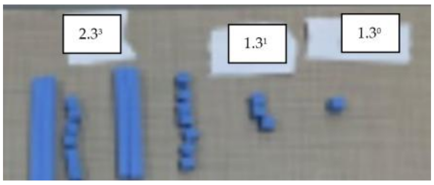


Figure 18. Proportional modeling through base ten blocks for $(10202)_3$- $(1121)_3$ (G2)

Utilizing non-proportional model in the base 3. G3 and G5 used non-proportional models. As shown in Figure 19, the participants matched specific place values with certain colors (Figure 19 a-c) or a certain number of connected unit cubes (Figure 19 d-f), or different colored connected unit cubes based on the base (Figure 19 g-k). In these representations where the trading process is not explicitly shown, the result of the operation are concretized as shown in these figures.

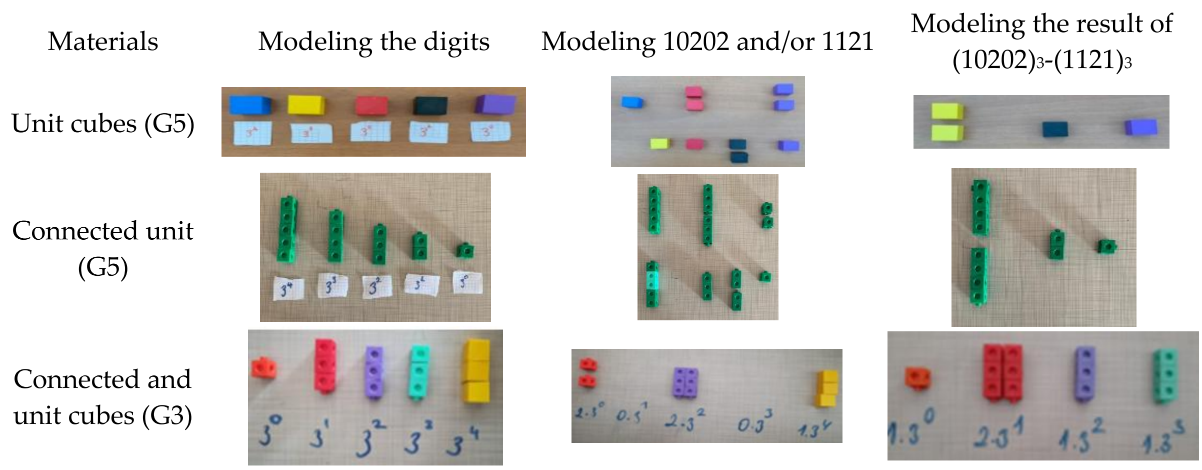


Figure 19. Non-proportional modeling through unit cubes and connected unit cubes

Utilizing mix model in the base 3. Only Group G4 used this model for subtraction in base 3. Accordingly, in Figure 20, it can be observed that students represented the ones place and threes place for numbers 3^0, 3^1 and 3^2, respectively, using an equal number of unit cubes and connected unit cubes, which correspond to the equivalents of 1, 3 and 9. For numbers 3^3 and 3^4, they accepted the use of base ten blocks corresponding to 100 and 1000, selecting a kind of non-proportional material to represent these numbers:

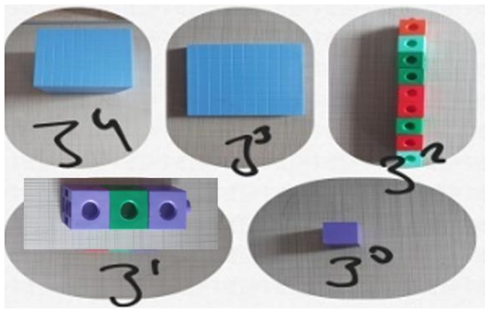
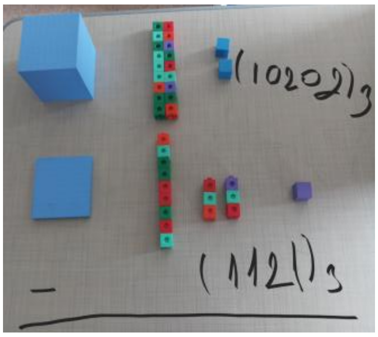
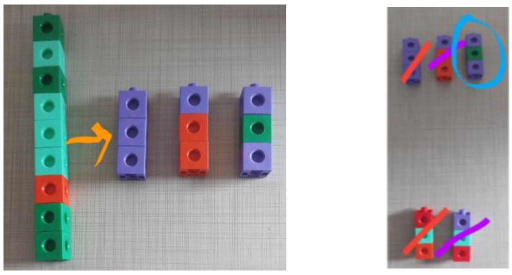


Figure 20. Assigning values of digits with mix model approach through base ten blocks and connected unit cubes (G4)

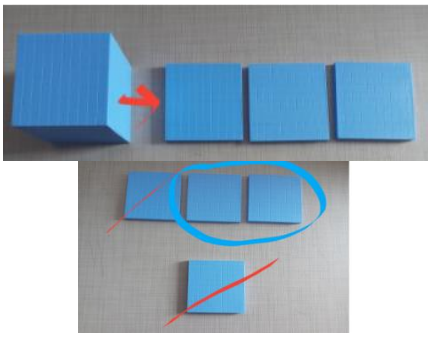
G4 wrote down the number representations and performed the necessary exchanges to carry out the subtraction operation, based on the representations shown in Figure 21.



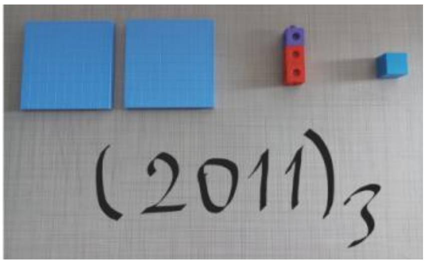
a. Representing 10202-1121 in Base 3



b. Transforming 3^2 to three 3^1 and making subtraction



c. Transforming 3^4 to three 3^3 and making subtraction



d. Modeling the result of $(10202)_3$-$(1121)_3$

Figure 21. Mix model approach for giving meaning to place value and 10202-1121 (G4)

Discussion and Conclusion

The aim of this study is to examine how prospective teachers conceptualize the place value concept in different number bases and how they utilize concrete materials in this process. The finding of the study that prospective middle school mathematics teachers (PMTs) used concrete materials to represent place value through both proportional and non-proportional models aligns with existing research on the importance of multiple representations in mathematics education, demonstrating that such representations support conceptual understanding. Multiple representations offer learners different perspectives on a mathematical concept, helping them develop a deeper understanding and make connections between different ideas (Lesh et al., 1987). In the context of place value, concrete materials can serve as a complementary, tangible representation that makes the concept more accessible to learners, in addition to more abstract models (Fuson & Briars, 1990; McNeil & Jarvin, 2007; Moyer, 2001). Accordingly, one of the main findings of the study is that prospective teachers not only used proportional and non-proportional models, as described in the literature (Van de Walle et al., 2018), but also structured their understanding of place value through a hybrid model. It can be said that the use of the hybrid model emerged as a particularly effective strategy for PMTs, as it allowed them to take advantage of the strengths of both proportional and non-proportional models for representing place value. It can be argued that PMTs were able to deepen their understanding of place value and develop a more flexible approach to problem-solving by starting with one model and then transitioning to another. This further demonstrates the ability of PMTs to adapt and refine their understanding of place value.

The findings of the study reveal those prospective teachers (PMTs) experienced difficulties in maintaining a diversity of materials across different base systems. This issue may stem from the PMTs' lack of sufficient experience in selecting and utilizing appropriate materials. Although concrete materials can enhance students' understanding of mathematical concepts, when these materials are not appropriately chosen the process of conceptual understanding can become complex and challenging for students (Carbonneau et al., 2013; Moyer, 2001). Although working with base systems posed challenges for PMTs, it can be argued that these challenges contributed to the expansion of their mental schemas regarding number systems. In this context, the inability of PMTs to maintain material diversity regardless of the base system highlights the difficulties they face in the process of constructing mathematical meaning. This situation underscores the need for PMTs to further develop their pedagogical content knowledge and material usage skills (Ball et al., 2008; Shulman, 1987).

Findings of the study showed that algebra tiles, wooden sticks, and geometry strips were the least preferred concrete materials by the groups for representing place value whereas unit cubes, connected unit cubes, and base ten blocks were the most frequently chosen materials. Among these materials, base ten blocks were the only material used across all different bases. The preference for base ten blocks can be attributed to their explicit representation of place value relationships. The different sizes of the blocks (units, rods, flats, etc.) directly correspond to the place values in the base-10 system, making it easier for learners to visualize and understand the concept (Hiebert & Wearne, 1992; Ross, 1989). Algebra tiles, wooden sticks, and geometry strips might have been less preferred due to their lack of an explicit connection to place value. These materials can be more versatile for various mathematical concepts (Clements, 2000), but they may not be as immediately recognizable as base ten blocks when representing place value. The versatility of base ten blocks is likely a key factor in their widespread use across different bases. These materials can be adapted to represent different place value systems, making them a valuable tool for teaching place value in various contexts (Fuson & Briars, 1990).

Algebra tiles were exclusively used in mixed modeling by only one group for addition (in base 10 and base 6) and subtraction (in base 3). In the addition operation conducted in base 10, proportional modeling was applied only between the hundreds and thousands place values, while non-proportional representation was utilized for the other place values. Similarly, in base 6, although a proportional approach was initially adopted, the multiplicative relationship was reflected in the manipulative only between the 6^2 and 6^3 place values. This suggests that the group initially intended to use proportional modeling but, due to the lack of appropriate materials, adopted algebra tiles as a proportional tool across different place values, thus creating a mixed model. Therefore, the use of a mixed model through algebra tiles likely originated from an attempt to implement proportional modeling. The geometric shapes of algebra tiles presented a challenge in representing the proportional relationship between the hundreds and thousands place values. While algebra tiles, with their square and rectangular shapes, are effective for emphasizing the concept of area and facilitating students' understanding of algebraic expressions by allowing them to visualize, manipulate, and relate concrete and symbolic representations (Çaylan, 2018; Okpube, 2016), they may not always clearly convey the proportional relationships between different place values.

Another way in which the mixed model was utilized was through the use of base ten blocks, connected unit cubes, and unit cubes in the base three system. Here, a proportional relationship was established between 3^0, 3^1, and 3^2, while a pseudo-proportional relationship was formed between 3^2, 3^3, and 3^4. Thus, similar to the previous use in the base six system, the mixed model draws from proportional modeling. Although there were enough interlocking unit cubes or unit cubes to represent proportions numerically, prospective teachers found it appropriate to use the geometric shapes of base ten blocks and algebra tiles to establish the multiplicative relationship between place values in two-dimensional and three-dimensional forms. Consequently, the design of base ten blocks may have paved the way for the adaptation to algebra tiles in representing place value.

Limitations and Recommendations

The sample of 24 junior prospective mathematics teachers might not be representative of the entire population of prospective mathematics teachers, potentially limiting the generalizability of the findings. The study was conducted in Turkey, and the findings might not be applicable to other regions or countries with different educational systems or cultural contexts. The fact that participants completed the activity in groups might affect the individual contributions and understanding of place value, making it difficult to assess individual learning outcomes. The four-week duration of the study might not have been sufficient to observe long-term changes in place value understanding. While the math laboratory provided a variety of concrete materials, the availability of these materials might have been limited, affecting the participants' choices and experiences.

By incorporating concrete materials into this activity, it may have provided prospective teachers with an opportunity for a deeper understanding of place value, similar to research studies that encourage the exploration of arithmetic operations in different bases using number line models (Roy, 2014) or contextual situations (McClain, 2009; Yackel et al., 2007). In line with these findings, activities that support prospective teachers' conceptual thinking in understanding and explaining mathematics without relying on memorization can be developed to facilitate their learning.

Considering the limitations and advantages of concrete materials, prospective teachers can be given the opportunity to express their thoughts using correct mathematical language. In this way, prospective teachers can be enabled to implement concrete material usage in their future classrooms with a deeper awareness. During this process, inquiries about the mathematical use of concrete materials can be conducted in mathematics education courses at the university level in teaching methods courses in mathematics education departments. To encourage prospective teachers for representing place value with more than one concrete material can be helpful for meaningful understanding. Considering that even adults seek help in making sense of unfamiliar situations without concrete representation, supporting the use of concrete materials by prospective teachers in mathematically focused topics can be beneficial.

We recommend that it is useful to provide concept-specific horizon content knowledge support to PMTs in order to support them in having a broader perspective on the mathematics they learn. We hope that this study, in which we reveal and support horizon content knowledge structuring as place value, will be an inspiration for knowledge structuring of other mathematical concepts. In this context, having prospective teachers explore the concept of place value through concrete materials not only facilitates their deep understanding of this crucial concept but also aids in developing strategies for effectively teaching it to students.

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