Education and Science

Vol 50 (2025) No 222 67-98

Original Article

Relations Between the Functions of Proof and Social and Socio-Mathematical Norms: Reflections from the KARİDE Model *

Tuğba Yulet Yılmaz [1], Nilüfer Yavuzsoy Köse [2]

Abstract

In addition to demonstrating the validity of mathematical propositions through proof, it also reveals functions such as explanation, discovery, systematization, and communication. It is essential to utilize all proof functions to transform it into a meaningful and profound mathematical activity in classrooms. Since mathematical proof and reasoning are social processes, teaching proof at the middle school level, particularly in demonstrating its importance and necessity, is influenced by certain norms. Therefore, teachers' consideration of proof's functions and the social and socio-mathematical norms supports students' reasoning processes. In this context, this study aims to reveal the functions of proof, social and socio-mathematical norms, and the relationship between these functions and norms within a learning environment that enables students to solve proof problems through interaction. The study participants comprised 7th-grade students attending a public middle school in the Central Anatolia Region. Since uncovering proof functions and establishing norms require long-term interaction in an authentic learning environment, the teaching experiment method was adopted in this study. The findings were presented by analyzing the video and audio recordings and worksheets obtained from the 12-week teaching process. The study results revealed that the classroom community developed shared norms regarding discussion, problem-solving, justification, and collaboration. These social and socio-mathematical norms guided students' participation in discussions and enhanced the quality of their contributions. Additionally, the study revealed that norms and functions of proof were intertwined in dialogues and that the norms across different themes and the functions of proof were interconnected. In the study, it was determined that the norms in the themes of discussion and collaboration mainly supported the communication function of proof, while the norms in the themes of

Keywords

Functions of proof
Mathematical proof
Mathematical reasoning
Classroom microculture
Social norms
Socio-mathematical norms

Article Info

Received: 05.25.2023
Accepted: 02.08.2025
Published Online: 04.25.2025

DOI: 10.15390/EB.2025.12938

* This article is derived from Tuğba Yulet Yılmaz's PhD dissertation entitled "Examining 7th grade students of the proving processes and the functions of proof in this process", conducted under the supervision of Nilüfer Yavuzsoy Köse.

[1] Van Yüzüncü Yıl University, Faculty of Education, Department of Mathematics and Science Education, Türkiye, tugbayulet@gmail.com
[2] Anadolu University, Faculty of Education, Department of Mathematics and Science Education, Türkiye, nyavuzsoy@anadolu.edu.tr

justification and problem-solving mainly supported the verification, explanation, discovery, and systematization functions of proof. It is recommended that studies focus on the social aspect of proof to demonstrate how social and socio-mathematical norms and the functions of proof support each other at different grade levels.

Introduction

Individuals who possess intense inquiry, research, and persuasion skills, who can construct and utilize knowledge based on their own experiences, and who are open to communication and collaboration have the potential to make a significant impact in shaping information and technology-driven societies. In this context, the ability to use mathematics to solve real-life problems, communicate in the language of mathematics, construct valid arguments, and engage in mathematical reasoning holds great significance in today's societies.

It is stated that mathematical reasoning forms the basis of mathematical competencies and that the lack of reasoning in mathematics teaching may lead to failure in the learning process (Battista, 2017), while mathematical proof is also defined as dependent on the reasoning process. For instance, Jahnke (2010) defines proof as a structure that involves reasoning based on induction and deduction, while the National Council of Teachers of Mathematics (2000) [NCTM] defines proof as a formal way of demonstrating specific forms of reasoning and justification. Similarly, Toker (2020) defines proof as a tool that can enhance mathematical thinking and discourse. It enables students to reason, justify their thoughts, make conjectures, draw inferences, test their ideas, and reach conclusions. Researchers who view mathematical proof as a process have utilized this process's stages in formulating definitions of proof (Baki, 2006; Dogan, 2015; Tall, 1998). When the studies of these researchers are examined, it is seen that the proof process consists of three intertwined stages: 1) *Understanding the Problem*, 2) *Making Conjectures*, and 3) *Justification and Proof*. Specifically, the proving process that can be used in school mathematics proof activities, along with the expected actions to emerge during this process, is presented in Figure 1:

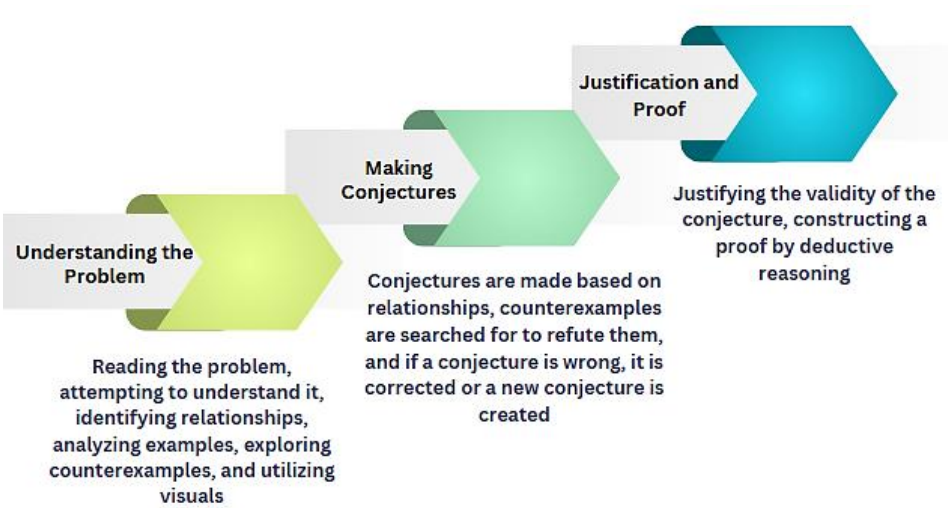


Figure 1. Stages of the Proof Process

Moreover, many researchers have proposed definitions of proof by emphasizing its socio-cultural aspects (Almeida, 2003; Fredriksdotter, Nor´en, & Bråting, 2022; Hersh, 2009; Jones & Herbst, 2012; Maher, 2009; Stylianides, 2007). For instance, Stylianides (2007) defines proof as a series of socially constructed claims whose validity depends on classroom norms. In this context, Stylianides emphasizes that for a mathematical argument to be considered proof, it should be built using statements accepted by the class without further justification, employ forms of argumentation appropriate to the class's level and conceptual accessibility, and be communicated through multiple representations. Fredriksdotter et al. (2022) view proof as communicative justification practices that consist of students explaining of how they solved a mathematical problem and the arguments they use to support or refute that solution. Ayala-Altamirano and Molina (2021) highlight the social aspects of justification and proof by emphasizing the social process by which mathematical knowledge is explained, verified, and systematized. These definitions emphasize that the concept of valid proof can be reached through the interaction among community members, underline that proof is a social activity used in mathematics classrooms to communicate students' reasoning, and thus stress the necessity of considering proof as a social process. It can be stated that the functions of proof reflect this social process.

The purpose a proof serves, its role in the classroom, and the meaning attributed to it by the person constructing the proof or interpreting an existing proof are explained through the functions of proof (Bell, 1976; De Villiers, 1990; Hanna, 2000). De Villiers (1990, 1999) states that proof has six functions: "verification," "explanation," "communication," "discovery," "systematization," and "intellectual challenge." The verification function refers to constructing a proof to demonstrate that a conjecture holds true in all cases, while the explanation function involves creating a proof to gain insight into why a conjecture is true (De Villiers, 1990). On the other hand, the communication function refers to the social interaction that involves sharing the results obtained through proof, discussing their validity and significance, and accepting or rejecting their correctness by different individuals (Herbst, Miyakawa, & Chazan, 2010). The discovery function refers to the discovery of new results and the generation of new ideas or conjectures, while the systematization function refers to the organization of results in a deductive system (De Villiers, 1990; Dennis, 2000; Hanna, 1983; Knuth, 2002). Intellectual challenge refers to proof in which a mathematician tests their intellectual endurance and creativity, and thus, the satisfaction derived from overcoming a mental challenge (De Villiers, 1999). On the other hand, Bartlo (2013) elaborated on five of De Villiers' functions of proof by defining them with their sub-functions, as shown in Table 1:

Table 1. Functions of Proof and Sub-functions (Bartlo, 2013)

Functions of Proof	Sub-functions of Proof	Indicators of Sub-functions
Verification	Conviction	Proof helps students overcome doubt by convincing them of the correctness of their conjectures. It allows them to validate their ideas and become autonomous learners.
	Confirmation	When the class accepts the correctness of a proven statement, its validity is confirmed, and the statement becomes a shared agreement within the classroom community.
Explanation	Insight	A proof provides insight into why a mathematical phenomenon is true and how it works.
	Consequences	When a theorem is used to prove another theorem, seeing its results helps students learn about the concepts used in the proof.
Communication	Form of discourse	Proof enables the communication of mathematical results between teacher and student or student and student.
	Forum for debate	The proof provides an opportunity for critical discussions in which students can correct their mistakes.
Discovery	Exploration	Proof facilitates the exploration of the consequences of conjectures, enabling the discovery of new results and allowing students to build upon their existing knowledge to learn new mathematical ideas.
	Analysis	By analyzing proof, more general conclusions can be discovered.
Systematization	Inconsistencies	Proof helps students see inconsistencies and learn mathematics by responding to counterexamples and arguments.
	Connections	Proof makes the relationships between concepts visible, facilitating the systematization of knowledge.
	Global perspective	Proof reveals the axiomatic structure underlying a topic, from which other related properties can be derived, thereby contributing to the systematization of knowledge.
	Application	Proof helps students generalize their solutions, enabling them to apply them to different problems.
	Alternative systems	Proof enables the development of new ways of thinking and the emergence of different deductive systems.

Experiencing the functions of proof both supports students' understanding of proof and their participation in proof practices (Hemmi, 2010) and enables the reflection of the role that proof plays in mathematical science in mathematics classrooms, thereby paving the way for proof to become a more meaningful activity in these settings (Hanna, 1995; Knuth, 2002). It is stated that proof is used as a tool to convince a community of the validity of a mathematical statement and that, to be persuasive, it must align with the norms of that community (Harel & Sowder 2007; Zaslavsky, Nickerson, Stylianides, Kidron, & Winicki-Landman, 2012). Similarly, for the classroom community to validate the correctness of a claim, the proof must conform to the classroom's social and socio-mathematical norms, and over time, proof influences and reshapes these norms.

A community is defined as a group of individuals sharing common cultural values, and each classroom is described as a community where teachers and students come together, share common goals, discuss, and shape their roles (Sekiguchi, 2006). At this point, a microculture emerges in the classroom environment through the mutual interactions between teachers and students, distinguishing that classroom from others (Cobb, Stephan, McClain, & Gravemeijer, 2011; Levenson, Tirosh, & Tsamir, 2009). In every classroom, the class members establish implicit and explicit understandings, which determine the behaviors concerning what each member does and values (Makar & Fielding-Wells, 2018). The mutual expectations that arise as a result of the communication and interaction of teachers and students in the classroom, the unwritten rules that describe the regularity in classroom activities, which are created jointly by teachers and students, which develop and change over time, are expressed as social norms (Cobb et al., 2011). These norms are constantly interpreted, transformed, or changed by teachers and students depending on the social interaction in the classroom. Socio-mathematical norms, on the other hand, are specific to the discipline of mathematics and are supported by the social norms of that community (Yackel & Cobb, 1996). Social and socio-mathematical norms are associated with student and teacher roles, beliefs about the general nature of mathematical activity, mathematical beliefs and values, and students' mathematical understanding (Cobb et al., 2011). Moreover, when implemented through inquiry-based and collaborative approaches in mathematics learning, it is known that these norms guide students' participation in mathematical discussions and enhance the quality of their contributions (Partanen & Kaasila, 2015). In this regard, norms play a crucial role in guiding students' participation in discussion-based classroom activities, encouraging them, and supporting their development.

When the studies are examined, it is seen that along with the studies that address the nature and development of social and socio-mathematical norms in the classroom (McClain & Cobb, 2001), there are studies that address the relationship between social and socio-mathematical norms, the role of the teacher, the strategies used, and the classroom environments based on discussion and inquiry in the creation and development of norms in the classroom (Ozdemir-Baki & Kilicoglu, 2023; Sekiguchi, 2006). There are also studies examining the socio-mathematical norms formed in a classroom where gifted students participate in problem-solving activities (Çakır, 2021; Çakır & Akkoç, 2024) and the socio-mathematical norms in a classroom with students with learning disabilities (Öksüz & Gürefe, 2021). For example, in Çakır's (2021) study, social and socio-mathematical norms were named under the headings of "Explanations," "Class Discussions," "Problem-Solving," "Problem Posing," "Valuing," "Inquiry,” and "Collaboration." The study addressed the expectations, awareness, and actions considered as evidence of norms through indicators defined for both teacher and student dimensions. Additionally, it has been observed that there are studies examining the teacher's pedagogical choices in discussion-based classroom environments designed to develop students' proof abilities, the socio-mathematical norms emerging in teacher and student actions, and emphasizing the social aspects of proof (Martin, McCrone, Bower, & Dindyal, 2005). This study differs from others as it deeply examines the proof functions that emerge during middle school students' proving processes, the norms that arise, and the relationships between proof functions and norms.

In proof-focused studies conducted at the primary and middle school levels, it is emphasized that proof should be addressed across all grade levels (Cervantes-Barraza, Moreno, & Rumsey, 2020; Mudaly, 2007; Rocha, 2019). These studies state that being introduced to formal proof during high school contradicts the nature of proof and mathematics. Instead, transitioning from less formal and more intuitive approaches to formal proof is necessary, and the functions of proof and its social aspects should be emphasized. In recent years, mathematics education research has focused much on students' communication skills in constructing new knowledge (Hanna & Knipping, 2020). With the growing interest in the social aspects of learning mathematics, how students' interaction patterns with each other and the teacher in classrooms transform into social and socio-mathematical norms is paving the way for new research (Partanen & Kaasila, 2015). However, it is known that most studies on students' mathematical argumentation and proofs focus on individual cognitive processes (Campbell, Boyle, & King, 2020). It is also stated that one of the reasons students develop negative attitudes toward proof and struggle with proving is the lack of sufficient information about the role and meaning of proof, as well as the neglect of its social aspect (Mudaly, 2007). In this context, this study ensured that students participated in proof activities and experienced the functions of proof in a classroom with a high level of social interaction. Indeed, it is essential to create environments where students can defend and support the arguments they develop, evaluate and try to refute others' arguments, understand the construction of mathematical concepts and processes by making mathematical justifications in these environments, and actively participate in this process (Chua, 2016; Yılmaz, 2021). Although the importance of proof and reasoning is emphasized in mathematics education, particularly in school mathematics, there appears to be a gap in the literature regarding middle school students' experiences with reasoning and proof, the meaning they attribute to mathematical proof, and the strategies they develop during the process of constructing mathematical proofs. Despite the emphasis that proof should be an integral part of mathematics education at all grade levels (Campbell et al., 2020; Fredriksdotter et al., 2022), it is noted that students are typically introduced to proof in high school, and the majority of studies on proof are conducted at the high school and undergraduate levels (Rocha, 2019). It is a significant necessity to discuss the function, importance, and limitations of proof in mathematics classrooms and to find the most effective ways to utilize proof in mathematics education (Hanna, 2000).

For all these reasons, in this study, the outcomes of students experiencing proof-making through group and classroom discussions in a classroom environment with high social interaction were wondered. For this purpose, the proof learning model "KARİDE" [Convince Yourself-Convince Your Friend-Convince Your Opponent-Evaluate], adapted from the stages of the proving process by Mason, Burton, and Stacey (2010) and supported by norms, was developed for use in proof activities. The proof was approached as a social process in the learning environment designed based on the KARİDE model. In this process, students were expected to encounter problem situations that were new to them, investigate the validity of these problems, and continuously justify their actions and statements while solving the problems. Furthermore, they were encouraged to discover the reasons behind their findings independently, communicate with one another using mathematical language, and carry out all these tasks within a specific systematic framework. In this learning environment, the study focused on the functions of proof that emerged, the social and socio-mathematical norms arising from the actions of students and teachers in the classroom microculture, and how the proof functions and norms supported each other, seeking answers to the following questions:

- What functions of proof emerge in the learning environment designed based on the KARİDE model?
- What social and socio-mathematical norms emerge in the learning environment designed based on the KARİDE model?
- What is the relationship between the proof functions and the norms in the learning environment designed based on the KARİDE model?

Method

Research Design

This study used a teaching experiment design to examine the functions of proof and the social and socio-mathematical norms that emerged during students' proving processes. A cycling approach is followed in the teaching experiment, consisting of consecutive teaching sessions, each consisting of a teaching agent (the teacher), one or more students, and the teaching recording to analyze the sessions (Steffe & Thomson, 2000). One of the most critical features of the teaching experiment, which helps to reveal new learning models, is the researcher assuming the role of the teacher (Steffe, 1991). In this study, a cyclical process consisting of consecutive teaching sessions was followed, the planning of each teaching session was guided by the analysis results of the previous session, and the researcher assumed the role of a teacher.

Participants

The study participants comprised 7th-grade students attending a public middle school in the Central Anatolia Region. In selecting the participants, qualitative research requirements were considered, and it was deemed necessary to choose participants who could provide access to rich data (Yalman & Uzungöz, 2021). For this reason, the criterion sampling method, one of the purposive sampling methods, was used to select participants. While purposive sampling is used to examine situations that contain rich information that will illuminate the problem in the research process (Meydan, 2021), criterion sampling is the study of situations that meet a set of predetermined criteria (Yıldırım & Şimşek, 2006). Some of the criteria determined include 7th-grade students being in the transition phase between informal and formal proof, one of the researchers teaching mathematics to this class during the study period, the class being suitable for group work, and the students being willing to participate in the research. In line with these criteria, the study participants comprised 31 students from the same class, including 15 girls and 16 boys. Before starting the study, students signed a voluntary consent form, and parents signed a research permission form. At the beginning of the teaching practices, the class was divided into four small groups: *Legendary Mathematicians, Citrus, Hardworking Bees,* and *Stars.* Since the social and socio-mathematical norms of each classroom differed according to their previous experiences, the researcher-teacher took into account the microculture of this classroom based on her observations before starting the research in initiating and maintaining norms.

Research Environment

A multi-group seating arrangement suitable for individual and group work was adopted in the classroom where the teaching was implemented. The video cameras used during the activities in the classroom (one in front and one in the back) were placed so that they could see the students, the researcher, and the blackboard and did not distract the students' attention. In addition, audio recorders were placed on the desks of the groups to prevent data loss.

Data Collection Tools and Data Collection Process

Ethical approval was obtained from Anadolu University with the decision dated 23.01.2019 and numbered 54380210-050.99 for this study. Permission for data collection within the scope of the research was granted by the Eskişehir Provincial Directorate of National Education with the decision dated 26.02.2019 and numbered 12377788-604.01.02-E.4196611.

In the study, data were obtained from audio and video recordings of teaching practices, worksheets used in activities during small group discussions, and notes written in the teacher's diary. Since this study focuses on the connection between norms in teaching practices and proof functions, the findings from the 12-week teaching process are presented. Before starting the 12-week teaching practices, a two-lesson introductory session on proof was conducted to outline the characteristics of a mathematically valid argument, argumentation methods (such as direct proof or providing counterexamples), and forms of argument representation (such as algebraic, visual, or verbal representations) (Stylianides, 2007). In this lesson, the researcher-teacher presented the argumentation methods considered valid in this class and explained the mathematical foundations on which mathematical explanations and justifications should be based to be acceptable. In this context, it was stated that providing examples has a limited role in proving the characteristics of an argument that would be considered valid without justification (such as definitions) and that examples could be used to understand the problem (proposition) during the proof process. Additionally, the importance of using strategic examples was highlighted; however, it was underlined that trial-and-error with examples would not be accepted as a valid proof method in this classroom. It was explained that providing a counterexample is sufficient to demonstrate that a proposition is false. Throughout the lesson, students were required to explain why their conclusions were correct. In this way, the teacher established social and socio-mathematical norms such as providing acceptable mathematical explanations and justifications, presenting valid mathematical proofs, rejecting experimental verification as a valid proof, and offering different justifications for problems.

Classroom activities were conducted over 24 lesson hours, scheduled as two 40-minute sessions (80 minutes) once a week for 12 weeks. In the selection of activities used in classroom sessions, the findings from the pilot study, the topics students were familiar with at the start of the implementation, the topics covered in previous academic years according to the curriculum, the topics included in the curriculum during the process, research on proof, expert opinions, and the alignment of prior and subsequent teaching sessions were taken into account. In this regard, activities were prepared considering the students' grade level and the pilot study's findings. The proof methods addressed within the scope of reasoning techniques that students could conceptually comprehend are presented in Appendix 1. Among the presented activities, three involve number problems and propositions that require students to construct direct proofs, one is a proposition that must be proven using a counterexample, four are geometry problems requiring direct proofs, two are number problems that require proof by exhaustion, one involves evaluating four different arguments regarding the validity of a proposition, and one is a pattern problem. While presenting their proofs, students were allowed to use verbal, visual, or algebraic representations, taking into account the level of 7th-grade students.

The stages of the KARİDE model, used in proof practices in the classroom and supported by norms, are presented in Figure 2:

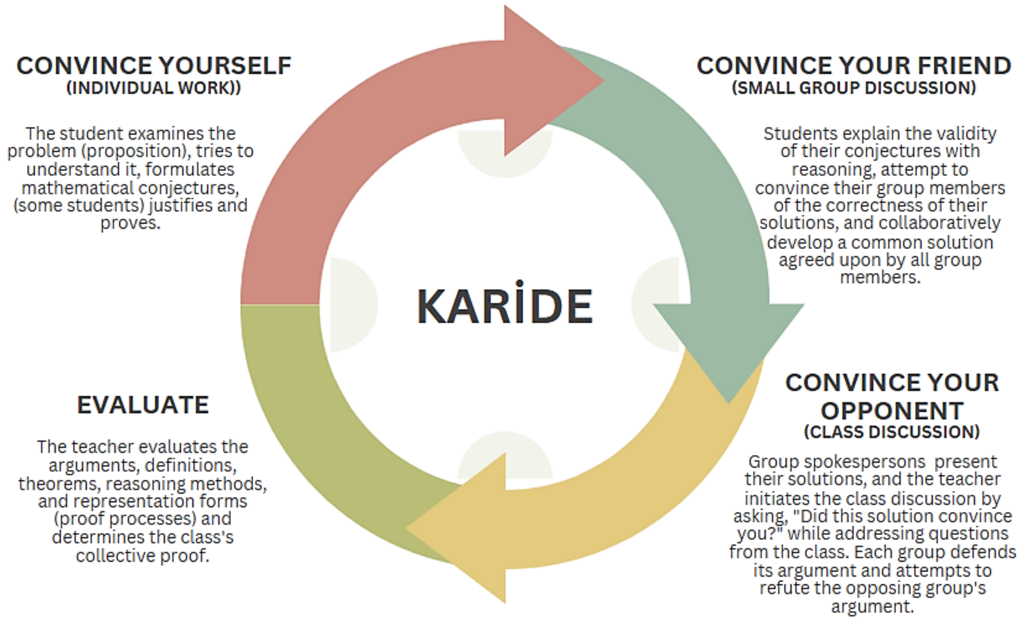


Figure 2. Stages of the KARİDE Model

The KARİDE model, as shown in Figure 2, consists of four stages: *a) Convince Yourself, b) Convince Your Friend, c) Convince Your Opponent,* and *d) Evaluate*. The *Convince yourself stage* is the stage where individual solutions are made. During this process, students must examine the given problem (or proposition), attempt to understand it and formulate mathematical conjectures. The actions performed in this stage may vary from one student to another. Within the given time, some students may use specific examples only to understand the problem, some may generalize by identifying standard features in the examples, and some students may complete the process and make proofs. *Convince your friend* stage is the phase where small group discussions take place. In this process, students are expected to explain the validity of the conjectures they developed during their work along with their reasoning to their group members, and they are also likely to reach a mathematical conclusion that the class will evaluate in the next stage. Throughout the *Convince your friend* stage, the teacher moves between groups, examining students' solution approaches and, when necessary, asking questions to help students deepen their thinking. It also encourages each group member to share their solutions with their peers, explain their ideas with reasoning, collaborate, ask each other about unclear points, listen to others until they finish speaking, and convince each other. Thus, it reinforces the formation of classroom norms. *Convince your opponent* stage is where the arguments developed by each group are discussed in the classroom to determine whether they constitute valid proof. In the *Convince your opponent* stage, the teacher asks students to pose questions to the person presenting the solution on the board, such as "How do you know this is correct?" "Can you justify why this claim is true?" "Can you explain why this is correct?" and "How can you be sure this is correct?" The teacher also guides students by asking similar questions and managing the classroom discussion. After each group explains the validity of its conjectures the reasoning, the teacher summarizes each group's solution so that the entire class can understand it. The teacher asks questions to clarify and make the presented ideas more understandable, repeating key points. Thus, it enables the presented explanations to be evaluated by other students. After each solution, the teacher repeats these actions, contributing to the continuity of class discussions. During the discussions, all questions from the class are received and answered; each group defends its arguments and attempts to refute the opposing group's arguments. The final stage of the model is the *Evaluation* stage. In this stage, the teacher evaluates the arguments developed by the groups, and the class decides upon a collective proof. During the implementation process, the stages of this model were followed in each lesson, with approximately 15 minutes allocated for the *Convince yourself* stage, 25 minutes for the *Convince your friend* stage, 30 minutes for the *Convince your opponent* stage, and 10 minutes for the *Evaluate* stage.

Data Analysis

Since the primary tools for collecting data on the teaching experiment were observation and video recordings of the teaching episodes, the collected data were analyzed following each teaching episode. With the findings obtained from the analysis, both the development of the students and the planning of the following teaching were made. Two critical levels of analysis were employed in the analysis of the teaching experiment: continuous analysis and retrospective analysis (Tanışlı & Yavuzsoy-Köse, 2013). In the ongoing research analysis, data obtained from teaching sessions were analyzed weekly while the teaching process was ongoing. In this context, the researchers reviewed the recorded videos and audio recordings at the end of each lesson during the teaching experiment to evaluate the previous teaching session, identify its shortcomings, and plan the next session accordingly. They thoroughly discussed the results obtained, their field notes, and observations of the classroom environment and tested assumptions in detail, making adjustments as deemed necessary. Accordingly, they made decisions by creating new conjectures for the next lesson. With this ongoing analysis, a teaching session included control of the predictions obtained from the previous session so that the research gained a non-static structure that could be changed and renewed according to the participants' learning due to the teaching experiment's structure (Steffe & Thomson, 2000). In the retrospective analysis, all collected data were examined holistically. The social and socio-mathematical norms that emerged during the study were defined based on descriptions commonly used in the mathematics education literature; in alignment with the theoretical framework of the research, closely related norms were consolidated under a single theme to ensure coherence. To facilitate the analysis and reveal the relationship between norms and functions, four norm themes, "Discussion," "Problem-Solving," "Justification," and "Collaboration," were developed, inspired by Çakır (2021) and a label (SN: Social Norm, SMN: Socio-mathematical Norm) was used for each norm. The norms under these themes were structured as follows: "It is expected that the classroom community shares their ideas (SN1)," "It is expected that different solutions to problems are proposed (SN2)," "It is expected that thoughts are explained and justified (SN3)," and "It is expected that solutions are collaboratively developed (SN4)."

Each norm under the themes was accepted as a fundamental norm initiating its sub-norms, and social and socio-mathematical sub-norms related to this fundamental norm were placed under it. For example, the social norm " It is expected that the classroom community shares their ideas. (SN1)" was adopted as the fundamental norm for initiating classroom discussions. The norms "Ensuring equal participation of students in the lesson (SN1a)," "Asking questions about unclear points during discussions (SN1b)," "Listening to the speaker until they finish speaking (SN1c)," "Sharing thoughts with group members and the class (SN1d)," "Presenting counterarguments by defending one's view (SN1e)," and "Refuting a peer's conjecture by defending one's view (SMN1e)" were identified as sub-norms associated with this fundamental norm. Although the norms were presented under different themes to facilitate analysis and provide a holistic view of the findings, the foundational norms are interrelated, and the sub-norms are also connected to other sub-norms. During coding, similar labels were used for social and socio-mathematical norms linked to the same foundational norm and supported each other's development. For example, the social norm "Presenting counterarguments by defending one's view" was labeled as "SN1e," while the socio-mathematical norm "Refuting a peer's conjecture by defending one's view," which supports the development of this social norm, was labeled as "SMN1e." In the study's findings, the sample dialogues that reflect the norms within a specific theme included only the norms of the theme being addressed. No norms outside those already present in the literature emerged during the study. Norms determined in teaching practices are presented in Table 2:

Table 2. Norms Determined in Teaching Practices

Theme	Norms	Sub-Norms
Discussion	It is expected that the classroom community shares their ideas (SN1)	Ensuring equal participation of students in the lesson (SN1a) Asking questions about unclear points during discussions (SN1b) Listening to the speaker until they finish speaking (SN1c) Sharing thoughts with group members and the class (SN1d) Presenting counterarguments by defending one's view (SN1e) Refuting a peer's conjectures by defending one's view (SMN1e)
Problem-Solving	It is expected that different solutions to problems are proposed (SN2)	Providing different justifications for problems (SMN2a) Using algebraic solution methods to make solutions more generalizable (SMN2b)
Justification	It is expected that thoughts are explained and justified (SN3)	Providing acceptable mathematical explanations and justifications (SMN3a) Presenting a valid mathematical proof (SMN3b) Rejecting experimental verification as a valid proof (SMN3c) Forming counterarguments by defending one's view (SN3d)
Collaboration	It is expected that solutions are collaboratively developed (SN4)	Reaching a common conclusion (SN4a) Forming a collective proof for the class (or group) (SMN4a)

Additionally, to analyze the functions of proof in instructional practices, Bartlo's (2013) framework of proof functions was used as a basis, and functions, sub-functions, and sub-function indicators were developed. For example, the "Verification" function was labeled as "D," and the sub-function "Conviction" was labeled as "D1." In this study, it was observed that two sub-functions present in the literature did not emerge during the research. It is thought that the reason why the sub-function "Exploration," which is one of the sub-functions of the "Discovery" function, and the sub-function "Alternative systems," which is one of the sub-functions of the "Systematization" function, did not emerge in this study is due to the grade level of the students and the proof problems selected by this grade level. Similarly, since this study was conducted at the middle school level, the "Intellectual Challenge" function from De Villiers' (1999) proof functions was not included. The functions and sub-functions that emerged in this study are presented in Table 3.

Table 3. Functions of Proof Determined in Teaching Practices

Functions of Proof	Sub-functions of Proof	Indicators of Sub-functions
Verification (D)	Conviction (D1)	Ensuring that the student is convinced of the correctness of his (her) conjecture through the solution he (she) has provided.
	Confirmation (D2)	Ensuring the confirmation of a statement by the class (group).
Explanation (A)	Insight (A1)	Ensuring the justification of why the conjecture is correct.
	Consequences (A2)	Ensuring that the results of the theorem used in the proof are observed and that information about the concepts used in the proof is obtained.
Communication (İ)	Form of discourse (İ1)	Ensuring communication between the teacher and the student or between students.
	Forum for debate (İ2)	Ensuring the creation of a forum for debates.
Discovery (K)	Analysis (K1)	Ensuring the discovery of more general results through analyzing a given proof.
Systematization (S)	Inconsistencies (S1)	Ensuring that inconsistencies are revealed through counterexamples.
	Connections (S2)	Ensuring the use of connections between concepts while constructing a proof.
	Application (S3)	Ensuring that solutions are generalized to be applied to a similar problem.

In the tables prepared to illustrate the relationships between norms and functions, the frequency of coding a norm and a function together in the in-class dialogs in the 12-week teaching experiment was considered. For example, the SMN3a norm and A1 function were coded together in the coding. However, some norms and functions needed to be coded together or coded more. Nevertheless, the weeks in which norms and functions were coded together are shown as upper indices in the tables. For example, the weeks in which the A2 function of proof and the SN1a social norm are coded together are indicated as $\checkmark^{3,5,8,10,11,12}$.

Validity and Reliability

At every stage of this research, the criteria determined by various researchers (Creswell, 2013; Merriam, 2018; Miles & Huberman, 1994; Yıldırım & Şimşek, 2006) were taken into consideration to ensure validity and reliability in qualitative studies. In the research, more than one data collection method was used, long-term interaction was established with the participants, expert review was used in data collection, data analysis, findings, discussion, and conclusion writing, and direct quotations were included in the findings. A pilot study was conducted before the research, and in-class activities were organized based on the pilot study results. The researchers coded all teaching activities separately, and the inter-coder reliability coefficient (Miles & Huberman, 1994) was calculated as 89%. Then, the similarities and differences in the codings made independently were compared, and the reasons were discussed. The revised version of the codings was presented to a different field expert, and the codings were finalized by making arrangements in line with their opinions.

Findings

In this section of the study, sample dialogues highlighting the norms and functions observed throughout the teaching experiment in the classroom along with the connections between these norms and functions, are presented.

Norms and Functions Emerging in Teaching Practices

In the learning environment designed according to the KARİDE model, it was observed that the researcher-teacher played an active role in initiating classroom norms, guided these norms and that students also contributed to the development of these norms throughout the teaching experiment. It was observed that the researcher-teacher reminded the students of the norms more frequently in the first weeks and asked them to adhere to them. In the following weeks, the students maintained the continuity of the norms without the needing to be reminded. To make the relationships between the norms themselves and their connections with functions more visible, the norms were interpreted under four themes: "Discussion," "Problem-Solving," "Justification," and "Collaboration" by data analysis framework.

Norms in the Discussion Theme and Their Connection with the Functions of Proof

The norms examined under the theme of "Discussion" and the functions supported by the development of these norms are presented in Table 4:

Table 4. Norms in the "Discussion" Theme and Their Connection with the Functions

		NORMS IN THE DISCUSSION THEME					
		It is expected that the classroom community shares their ideas (SN1)					
Functions of Proof		SN1a	SN1b	SN1c	SN1d	SN1e	SMN1e
Verification (D)	D1	**✓All weeks**	**✓All weeks**	**✓All weeks**	**✓All weeks**	**✓All weeks**	**✓All weeks**
	D2	**✓All weeks**	**✓All weeks**	**✓All weeks**	**✓All weeks**	**✓All weeks**	**✓All weeks**
Explanation	A1	**✓All weeks**	**✓All weeks**	**✓All weeks**	**✓All weeks**	**✓All weeks**	**✓All weeks**
(A)	A2	**✓3,5,8,10,11,12**		**✓3,5,8,10,11,12**	**✓3,5,8,10,11,12**		
Communication	İ1	**✓All weeks**	**✓All weeks**	**✓All weeks**	**✓All weeks**	**✓All weeks**	**✓All weeks**
(İ)	İ2	**✓All weeks**	**✓All weeks**	**✓All weeks**	**✓All weeks**	**✓All weeks**	**✓All weeks**
Discovery (K)	K1			**✓10**	**✓10**		
Systematization	S1	**✓All weeks**	**✓All weeks**	**✓All weeks**	**✓All weeks**	**✓All weeks**	**✓All weeks**
(S)	S2			**✓3,5,8,10,11**		**✓3,5,8,10,11**	**✓3,5,8,10,11**
	S3			**✓4,5,7,8,10,12**			

As seen in Table 4, the norm "It is expected that the classroom community shares their ideas (SN1)" is a fundamental norm under this theme, initiating and organizing classroom discussions while supporting other social and socio-mathematical norms. The different norms under this theme have been identified as follows: "Ensuring equal participation of students in the lesson (SN1a)," "Asking questions about unclear points during discussions (SN1b)," "Listening to the speaker until they finish speaking (SN1c)," "Sharing thoughts with group members and the class (SN1d)," "Presenting counterarguments by defending one's view (SN1e)," and "Refuting a peer's conjecture by defending one's view (SMN1e)." To reflect the norms under this theme, a cross-section of the small group discussions of the Legendary Mathematicians group during the third week, involving a geometry problem requiring direct proof (Appendix 1), is presented below as an example.

> ***Teacher:*** *Now the convince your friend phase has started. You should share the solution you worked on individually with your group members. Please explain your ideas along with the reasoning behind them. Ask each other about anything you do not understand, and convince one another that your solution is correct. You will now work on creating a common solution for your group, so please listen to your friends until they finish speaking. Then, we will move on to the*

"convince your opponent" phase. By the way, I want a different spokesperson to be on the board every week. (SN1,SN1a,SN1b,SN1c,SN1d,D1,D2,A1,İ1,İ2)

…

Student 1: *Did you measure it? Mine came out weird; for example, in the question, it says 6 cm here, but when I measured it, it came out something like 3.* (SN1,SN1a,SN1b,SN1c,İ1,İ2)

Student 2: *I measured more than 3 and then I stopped measuring. I did it my own way and found D to be 60º. Everyone, tell me what you did.* (SN1,SN1a,SN1c,SN1d,D1,İ1,İ2)

…

Student 2: *Yes, that's already an equilateral triangle. How did you solve it?* (SN1,SN1a,SN1c,SN1d,D1,D2,İ1,İ2)

Student 3: *I found angle D to be 24° and angle DAC to be 36°.*(SN1,SN1a,SN1c,SN1d,D1,İ1,İ2)

Student 2: *How did you find it like that? Now look, this is already a right triangle. The teacher gave 60º here and 30º here. This is the right angle, 180º. If we subtract 60 from 180, we get 120º, so what you said is refuted.* (SN1,SN1a,SN1b,SN1c,SN1d,SN1e,SMN1e,D1,A1,İ1,İ2,S1,S2)

Student 3: *I got it mixed up.* (İ1,İ2)

...

Student 1: *Do you remember how we used to do something with the base and height? In an equilateral triangle, if this side is 6, DC also seems to be 6 to me. When you draw the height, it divides the triangle right in the middle. I just pulled it now; with a base of 6, it became 3 cm and 3 cm.* (SN1,SN1a,SN1c,SN1d,D1,A1,A2,İ1,İ2,S2)

…

Student 2: *Now look, I just noticed something. It is an isosceles triangle, so these sides must be equal, which means they are both 6.* (SN1,SN1a,SN1c,SN1d,D1,A1,A2,İ1,İ2,S2)

Student 3: *So, are we saying it's 6 cm because this is an isosceles triangle?* (SN1,SN1a,SN1b,İ1,İ2)

Student 2: *Yes, actually, you find the isosceles triangle like this: this angle is already given as 30° in the question, and we also calculated angle D as 30º. Since both are 30º, it's isosceles. That's why these sides are equal. Do you remember last year when Gözde teacher drew arrows across the isosceles triangle (to show the equality of the sides opposite the equal angles)? Those sides were equal. Here, if we draw arrows for the 30º angles since AC is 6 cm, DC must also be 6 cm. Is anyone not convinced?* (SN1,SN1a,SN1c,SN1d,D1,D2,A1,A2,İ1,İ2,S2)

Student 3: *Oh yes, how did I not see that? It's so simple. Now, let's explain it again.* (D2,İ1,İ2)

…

The example dialogue provided above shows that throughout the group discussion, students tried to understand each other, asked each other about unclear points, and tried to convince each other. It is also seen that each student in the group is aware of the need to share their ideas and participate in the discussion, and the speaker is listened to until they finish speaking. These situations indicate that the group has agreed on the norms under the discussion theme (SN1,SN1a,SN1b,SN1c,SN1d). In the discussion, it was observed that one student determined the measure of angle D as 24° and the measure of angle DAC as 36°, upon which a groupmate presented a counterargument by defending their view and refuting their peer's conjecture (SN1e,SMN1e). This dialogue is also associated with the sub-function of the systematizing function of proof to reveal inconsistencies and relationships (S1,S2). It is observed that, through the proof they constructed, the students were convinced that the measure of angle D is 30° and the length of line segment DC is 6 cm without needing any external authority (D1,D2). Furthermore, a student made the conjecture that triangle ADC is an isosceles triangle and explained the validity of this conjecture along with its reasoning to their peers (A1). It is also observed that the students used the properties and theorems they knew about triangles in their proofs and recognized their consequences (A2). As observed, the fact that students are convinced of the validity of their conjectures through their solution and the need to justify why their conjectures are correct for the solution to be validated demonstrate that proof's verification and explanation functions support each other.

Furthermore, the fact that the students in the class reached a common agreement during the teaching experiment that they should share their ideas (SN1) reveals their form of discourse, creates a forum for debate (İ1, İ2), and highlights the connection between the norms under the discussion theme and the communication function of proof. Additionally, throughout the teaching experiment, it was observed that for the students to be convinced of the proof's validity and confirm its validity (D), it was necessary to justify why it was correct (A). The emergence of these functions also depends on the students sharing their ideas and convincing one another. This situation highlights the connection between the norms under the discussion theme and the verification, explanation, and communication functions of proof. The findings from the 12-week teaching experiment demonstrate that the norms under the discussion theme and the functions of proof are intertwined and interconnected within the dialogues.

Norms in the Problem-Solving Theme and Their Connection with the Functions of Proof

The norms examined under the "Problem-solving" theme and the functions supported by the development of these norms are presented in Table 5:

Table 5. Norms in the "Problem-Solving" Theme and Their Connection with the Functions

		NORMS IN THE PROBLEM-SOLVING THEME	
		It is expected that different solutions to problems are proposed (SN2)	
Functions of Proof		SMN2a	SMN2b
Verification (D)	D1	**✓All weeks**	**✓All weeks**
	D2	**✓All weeks**	**✓All weeks**
Explanation (A)	A1	**✓All weeks**	**✓All weeks**
	A2	**✓3,5,8,10,11,12**	
Communication (İ)	İ1	**✓All weeks**	**✓All weeks**
	İ2	**✓All weeks**	**✓All weeks**
Discovery (K)	K1	**✓10**	
Systematization (S)	S1		
	S2	**✓3,5,8,10,11**	**✓3,5,8,10,11**
	S3	**✓4,5,7,8,10,12**	

As presented in Table 5, the norm "It is expected that different solutions to problems are proposed (SN2)" is considered the fundamental norm under this theme, as it initiates and supports the socio-mathematical norms of "Providing different justifications for problems (SMN2a)," and "Using algebraic solution methods to make solutions more generalizable (SMN2b)." In the first weeks of the proof activities, the researcher-teacher facilitated the use of different representations in the classroom to help students recognize that there could be multiple solution methods and allowed for the validity of the result to be demonstrated through various justifications and approaches. In response to the teacher's expectation of providing different solutions to problems, students, being aware of this expectation, attempted to offer different justifications for problems even when not explicitly requested by the teacher and encouraged one another to use algebraic solution methods. These findings indicate that the group agreed on the norms under the problem-solving theme (SN2,SMN2a,SMN2b). A cross-section of the small group discussions of the Legendary Mathematicians group during the 12th-week pattern problem (Appendix 1) and the solutions provided by the students are presented below as an example of classroom dialogue reflecting the norms under this theme:

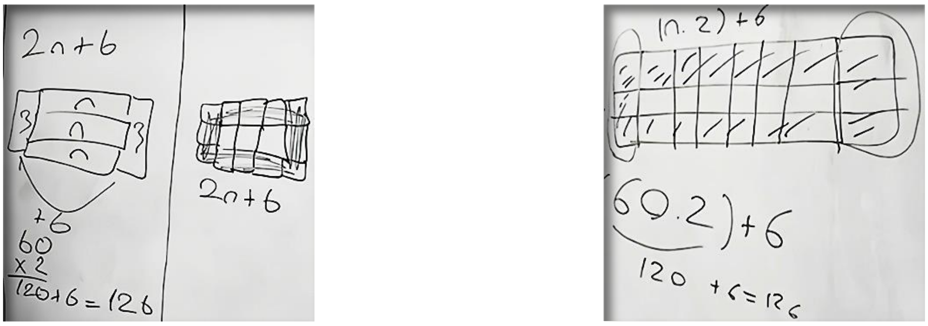


Image 1. Small Group Discussion of the Legendary Mathematicians Group in the Twelfth Week

> ***Student 1:*** *Here's what I did. I separated the three tiles on each side. I said there's as much gray above and below the white tiles as the white tiles themselves. That's why I first wrote 2x. Then, since three more tiles were needed to cover the sides, I added +6.* (D1,D2,A1,İ1,İ2)
>
> ***Student 2:*** *Mine is the same as yours; we have the same formula. This part will be n, meaning there will be n both at the top and bottom and 3 on each side, so 3 and 3 make 6. That's 2x+6 or 2n+6. When it's 60, it comes out to 126 anyway.* (D1,D2,A1,İ1,İ2)
>
> …
>
> ***Student 1:*** *But look, we're adding +6, so what Nefise did seems strange.*
>
> ***Student 3:*** *Yours is correct, but mine is also accurate. Do you know why? Look, for however many white tiles there are, we take two more than that, which accounts for the grays above and below. Then, we add the remaining 2.* (SN2,SMN2a,D1,D2,A1,İ1,İ2)
>
> ***Student 4:*** *There are two correct perspectives here. We examined it by separating the sides, while you considered the entire top and bottom.* (SN2,SMN2a,D1,D2,A1,İ1,İ2)
>
> ***Student 3:*** *Yes, you divided it into parts like that, while I divided it this way. You looked at it by directly removing the 6 tiles.* (SN2,SMN2a,D1,D2,A1,İ1,İ2)
>
> ***Student 2:*** *Actually, I just realized that Nefise's method and ours are the same, and both are correct. Because she did (n+2)×2, where the 2 multiplies both terms, making it 2n+4, and then she adds 2, so it becomes 2n+6. That means both are correct.* (SN2,SMN2a,D1,D2,A1,İ1,İ2)
>
> …

In the discussion, it is observed that the students analyzed the shape in various ways and discovered different generalization methods. In the initial part of the discussion, although there was disagreement among the students, it was determined that when they analyzed the shape differently, they found the general rule equivalent. As observed in the discussion, the student's pursuit of different solutions and their effort to justify how the methods they used in their solutions differed from one another indicates that the classroom community accepted this norm (SN2,SMN2a). At the same time, the emergence of these norms in the students' actions, along with their attempts to convince their group members of the validity of their conjectures by providing justifications for how their solutions differed from one another, points to the verification, explanation, and communication functions of proof (D1,D2,A1,İ1,İ2).

Additionally, the social norm "It is expected that different solutions to problems are proposed (SN2)" initiates and supports the development of the socio-mathematical norm "Using algebraic solution methods to make solutions more generalizable (SMN2b)." Below is the small group discussion of the Legendary Mathematicians group during the 10th week of the teaching experiment, where they were required to perform a direct proof for a geometry problem (Appendix 1), and their solutions, shown in Image 2, reflect this norm. In this discussion, a group member told Student 2 they were using an example to prove the statement. The student explained that these numbers were used solely to show the equality of the angle measures and that they would reach the same result regardless of which angle they used. These findings indicate that the student was using a generalizing example. Furthermore, it was observed that, despite their peers presenting counterarguments to the student who used numerical values to demonstrate the equality of side lengths. The student employed algebraic methods to make their solutions more generalizable and successfully convinced their peers (SMN2b,D1,D2,A1,A2,İ1,İ2,S1,S2).

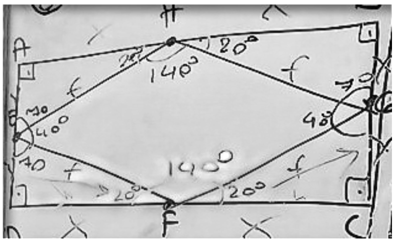


Image 2. Small Group Discussion of the Legendary Mathematicians Group in the Tenth Week

…

Student 2: *Sure, you can write any number you want. I just showed it so you can see which angle equals which. Take any numbers that add up to 90. For example, if we assign 5 to one of these parts. It's already given in the question that these are equal, so this part will also be 5. Since this is a rectangle, the opposite sides will also be 5 and 5. So, if this side is x, then this side is also x. If this side is y, then the opposite side is also y. It's already given in the question. I think if this side of the inner quadrilateral is f, then the opposite side is also f.* (SN2,SMN2a,SMN2b,D1,A1,A2,İ1,İ2,S2)

…

Student 1: *Wouldn't this count as a trial-and-error method?*

Student 2: *No, it wouldn't count. We only used numbers to show which angles are equal…*

Student 4: *But you assigned numbers again.*

Student 2: *No, I assigned numbers to show you that it's a rhombus... because all these sides are the same in the triangles. Look, I won't assign numbers this time for you. Now, all of these are y, and all of these are x because this is a rectangle. These triangles are all congruent. Since this angle is 20, this one is also 20, and the remaining angle is 140, and the opposite angles are the same. These angles here are 70. Therefore, the remaining sides of these triangles must also be equal. We've written the angles, and this is a rhombus*. (SMN2a,SMN2b,D1,A1,A2,İ1,İ2,S1,S2)

Student 1: *Alright, I'm convinced*.(D2)

…

In the dialogue above, it is observed that the students proved that quadrilateral EFGH is a rhombus by utilizing the congruence of the triangles formed at the corners of the rectangle. Through their proof, they convinced themselves of the validity of their conjectures without requiring any external authority. The group approved the proof, and the students explained the correctness of their conjectures along with the underlying reasons (D1,D2,A1,A2,İ1,İ2,S2). Additionally, in the same week, the Citrus group demonstrated the congruence of the triangles by utilizing the symmetry line of the rectangle with a different solution method, as presented in Image 3 and the class also approved this solution:

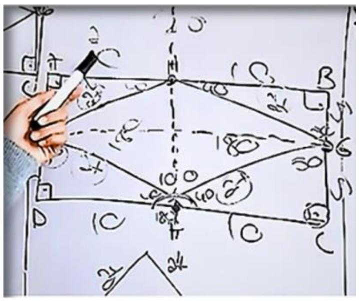
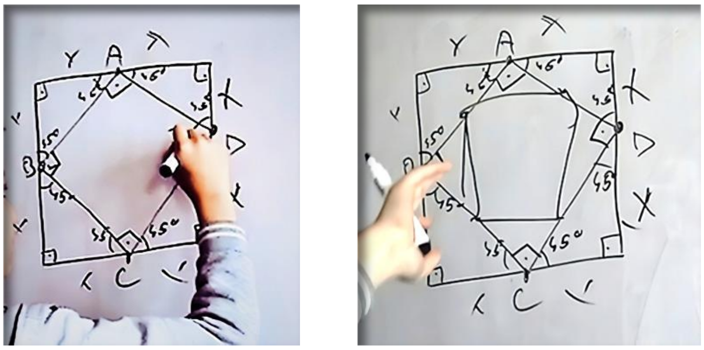


Image 3. Class Discussion of the Citrus Group in the Tenth Week

It is observed that the students utilized various previously learned concepts, such as the definition of a right triangle, the definition of a rectangle, the definition of a right angle, the properties of a rhombus, and the symmetry line, by connecting them in this proof (S2). In the proof activities, students were enabled to gain insight into why a mathematical statement is true and to acquire more profound knowledge about the properties they used in the proofs by observing their consequences (A1,A2).

In the 10th week evaluation phase, the teacher asked, "*Now, if this is understood, I will ask one more question. What can you say about the quadrilateral formed when we connect the midpoints of the sides of a square?*" Many students offered different justifications. As presented in Image 4-a, a student demonstrated that the triangles formed at the corners are isosceles and proved that the resulting quadrilateral is a square. In this problem, students generalized the solution method they found and used it in different problems they encountered, which served the *Application* (S3) sub-function of the systematization function:



a b

Image 4. Evaluation Phase in the Tenth Week

After the teacher expanded the problem, it was observed that one student, as presented in Image 4-b, generalized that the quadrilateral formed at the center by connecting the midpoints of the sides of the resulting square would also be a square. Different justifications were presented when the teacher gave the floor to the class. The dialogue is provided below:

Student 1: *Teacher, I want to ask you something: if we were to connect the midpoints of the sides of this square as well, would it also form a square? (K1)*

Teacher: *Yes, what do you say? Would that be a square, too, he says?*

Student 2: *It would be a square. Since the previous one was also a square derived from within a square, this one would also be a square. (SN2,SMN2a,D1,D2,A1,A2,İ1,İ2,K1,S2)*

Student 3: *It will be square because the triangles on the sides will be isosceles again. (SN2, SMN2a,D1,D2,A1,A2,İ1,İ2,K1,S2)*

Teacher: *This would go on infinitely, wouldn't it? Even if we draw inside it again and connect the midpoints of the sides, it would also form a square.*

The analysis of proof facilitated the discovery of more general results and served the discovery function through the sub-function of *Analysis* (K1). In this class, the norm of presenting different solutions to problems has fostered a rich discussion environment by encouraging the application of students' generalized solutions to similar problems (S3) and making discoveries (K1).

Throughout the teaching experiment, students' ability to provide different justifications for problems (SMN2a) required them to reason why their conjectures were correct (A), supporting the explanation function of proof. At the same time, it supported the verification function of proof by enabling the class to be convinced of the correctness of these conjectures (D). Additionally, students' efforts to establish connections between different mathematical concepts (S2) while presenting their solutions to provide various justifications supported the systematization function. The emergence of these functions also depended on students sharing their ideas and communicating (İ).

Norms in the Justification Theme and Their Connection with the Functions of Proof

The norms examined under the "Justification" theme and the functions supported by the development of these norms are presented in Table 6:

Table 6. Norms in the "Justification" Theme and Their Connection with the Functions

		NORMS IN THE JUSTIFICATION THEME It is expected that thoughts are explained and justified (SN3)			
Functions of Proof		SMN3a	SMN3b	SMN3c	SN3d
Verification (D)	D1	**✓All weeks**	**✓All weeks**	**✓All weeks**	**✓All weeks**
	D2	**✓All weeks**	**✓All weeks**	**✓All weeks**	**✓All weeks**
Explanation (A)	A1	**✓All weeks**	**✓All weeks**	**✓All weeks**	**✓All weeks**
	A2	**✓3,5,8,10,11,12**	**✓3,5,8,10,11,12**		
Communication (İ)	İ1	**✓All weeks**	**✓All weeks**	**✓All weeks**	**✓All weeks**
	İ2	**✓All weeks**	**✓All weeks**	**✓All weeks**	**✓All weeks**
Discovery (K)	K1	**✓10**	**✓10**		
Systematization (S)	S1	**✓All weeks**	**✓All weeks**	**✓All weeks**	**✓All weeks**
	S2	**✓3,5,8,10,11**	**✓3,5,8,10,11**		
	S3	**✓4,5,7,8,10,12**	**✓4,5,7,8,10,12**		

As presented in Table 6, the norm " It is expected that thoughts are explained and justified (SN3)" is considered the fundamental norm under this theme, as it initiates and supports the socio-mathematical norms of "Providing acceptable mathematical explanations and justifications (SMN3a)," "Presenting a valid mathematical proof (SMN3b)," "Rejecting experimental verification as a valid proof (SMN3c)," and "Forming counterarguments by defending one's view (SN3d)."

To reflect the norms within this theme, the class discussion from Week 5 involving the Stars group, where they were required to directly prove a geometry problem (Appendix 1), is presented below. It was determined that the group correctly divided the polygons into triangles, as shown in Image 5, and calculated the interior angle measures of the polygons using the sum of the interior angles of the triangles. A cross-section of the class discussions of the Stars group is presented below:

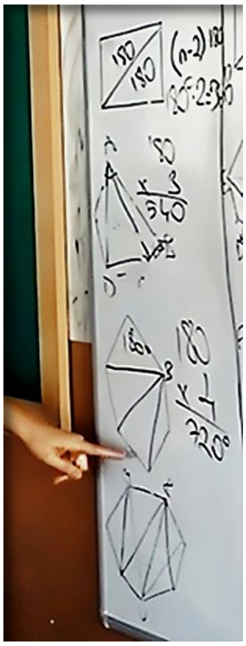


Image 5. Class Discussion of the Stars Group in the Fifth Week

Group Spokesperson: *Teacher, we divided this rectangle into triangles, resulting in two triangles; the sum of the angles in a triangle is 180°. We added 180 and 180, which gave us 360. So, teacher, when we did the same for others, we discovered something: if you take the number of vertices, subtract two, and multiply it by 180, you get the sum of the interior angles. For example, a 30-sided polygon has 30 vertices; subtracting two gives 28, and multiplying 28 by 180 gives the result. For an n-sided polygon, it becomes (n-2) × 180.* (SN3,SMN3a,SMN3b,D1,D2,A1,A2,İ1,İ2,S2)

Teacher: *Why did you divide it into triangles? Where did you come up with it?*

Group Spokesperson: *Because we knew that the angles of a triangle add up to 180. At first, I divided the pentagon into a triangle and a quadrilateral. The triangle's angles were 180, and the quadrilaterals were 360, which added up to 540. But then I thought, for example, if it were a decagon, and I divided it into a triangle and a nonagon, I wouldn't know the angles of the nonagon. So, I decided to divide everything into triangles.* (SN3,SMN3a,SMN3b,D1,D2,A1,A2,İ1,İ2,S2)

…

Student 2: *So, how can you be sure that (n-2).180 is correct for all polygons?*

Group Spokesperson: *In all polygons, the number of triangles formed is always equal to two less than the number of sides. Since the sum of the interior angles of a triangle is 180, we multiply that by 180 to get the result.* (SN3,SMN3a,SMN3b,D1,D2,A1,A2,İ1,İ2,S2)

In this discussion, it was observed that the spokesperson grounded their explanations and justifications on mathematical principles in response to the teacher asking why they divided the shape into triangles and Student 2 requesting justification for why their generalization is valid for all polygons (SN3,SMN3a). Aware of the expectation for justification, students' acceptable mathematical explanations regarding their solutions and the concepts they used support the proof's explanation (A) function. At the same time, students' efforts to establish connections between the properties of different polygons while supporting justifications the connections sub-function (S2) of the systematization function of proof. It is observed that both the group and the class verified through this proof that the sum of the interior angle measures of a polygon is (n-2).180 (D1,D2). Furthermore, it emerges as an established norm in small group and class discussions within this classroom that presenting a valid mathematical proof is necessary, experimental verification is not accepted as a valid proof, and students present counterviews while defending their view (SMN3a,SMN3b,SMN3c,SN3d). For instance, it was observed that a student from the Citrus group attempted to find a multiplicative relationship between the sums of the interior angle measures of polygons for the same problem, but the student's group members did not accept this numerical approach. The student's solution in the group discussion is as follows:

…

Student 1: *Look, I found something. I think we should present this on the board as well. Now, with the formula you saw, you calculated the sum of the angles as 360 for a quadrilateral, 540 for a pentagon, 720 for a hexagon, and 900 for a heptagon, right? You're always adding 180. I wrote them all down in order. Now, for the quadrilateral, it's 4 times 90; for the pentagon, it's 6 times 90; for the hexagon, it's 8 times 90. So, it increases by 2 times each time.*

Student 2: *We already found something general. You're still working with examples.* (SMN3c,SN3d)

…

In the same group, a student measured the interior angles of a quadrilateral using a protractor, finding each angle to be 90°, and suggested that the same method could be used for other polygons; however, their peers rejected this approach. Similarly, another student assigned values to the interior angles of polygons based on visual perception, which their peers also dismissed. This situation shows that, in the classroom, proof must conform to the social and socio-mathematical norms of the class to confirm the validity of a claim. Indeed, since experimental verification is not accepted as a valid proof method in this classroom, its validity was not confirmed, and counterviews were formed (SMN3c,SN3d). It was observed throughout the entire teaching experiment that the class community reached a common agreement on the explanation and justification of ideas (SN3), and this norm revealed the insight (A1) sub-function of the explanation function of proof. At the same time, it was observed that for a solution or statement to be convincing and its validity to be confirmed (D1, D2), the explanations provided must be acceptable and align with the norms within the justification theme, which revealed the verification function of proof. The emergence of acceptable mathematical explanations and valid proofs is influenced by both the justifications provided by students to each other and the feedback and evaluations given by the teacher (SMN3a,SMN3b). Indeed, the emergence of these norms and functions depends on students sharing their ideas and communicating with each other and their teachers (İ).

Norms in the Collaboration Theme and Their Connection with the Functions of Proof

The norms examined under the "Collaboration" theme and the functions supported by the development of these norms are presented in Table 7:

Table 7. Norms in the "Collaboration" Theme and Their Connection with the Functions

		NORMS IN THE COLLABORATION THEME It is expected that solutions are collaboratively developed (SN4)	
Functions of Proof		SN4a	SMN4a
Verification (D)	D1	**✓All weeks**	**✓All weeks**
	D2	**✓All weeks**	**✓All weeks**
Explanation (A)	A1	**✓All weeks**	**✓All weeks**
	A2	**✓3,5,8,10,11,12**	**✓3,5,8,10,11,12**
Communication (İ)	İ1	**✓All weeks**	**✓All weeks**
	İ2	**✓All weeks**	**✓All weeks**
Discovery (K)	K1		
Systematization (S)	S1	**✓All weeks**	**✓All weeks**
	S2	**✓3,5,8,10,11**	**✓3,5,8,10,11**
	S3		

As presented in Table 7, the norm "It is expected that solutions are collaboratively developed (SN4)" is considered the fundamental norm under this theme, as it initiates and supports the socio-mathematical norms of “Reaching a common conclusion (SN4a),” “Forming a collective proof for the class (or group) (SMN4a).” Since the KARİDE model is designed to facilitate students' collaboration in small group activities, producing solutions together, and constructing a common proof during classroom discussions, the norms under the theme of collaboration are explicitly evident throughout all dialogues. Developing of a common understanding among students regarding the necessity of collaborating and producing solutions together (SN4) facilitated communication among them and created a forum for debate (İ1,İ2). This situation demonstrates the connection between the norms under the theme of collaboration and the communication function of proof. Throughout all weeks of the teaching experiment, it was observed that students who could not solve problems individually could produce solutions together with their peers in small groups, striving to develop a common solution agreed upon by all group members (SN4a,SMN4a). For small groups to reach a common conclusion, it was necessary for all members to be convinced of the validity of the conjectures and for the group to confirm their correctness (D1,D2). This situation demonstrates that the norms under the theme of collaboration support the verification function of proof. In the process of reaching a common conclusion (SN4a), incorrect conjectures were refuted with counterexamples (S1), objections were raised against experimental validations, and students were required to provide mathematical explanations (A) to demonstrate the validity of the conjectures. This situation indicates that the norms under the collaboration theme support proof's explanations and systematization functions.

Conclusion, Discussion, and Recommendations

This study aims to identify the social and socio-mathematical norms arising from student and teacher actions within the classroom microculture, the functions of proofs, and the relationship between proof functions and norms in a learning environment designed according to the KARİDE model. The study results showed that most of the class members adopted the norms of discussion, problem-solving, justification, and collaboration as the elements of the interaction structure in the lesson sessions created according to the KARİDE model in the 12-week teaching experiment. These norms emerged prominently in the dialogues in the discussions. The results show that these norms are interrelated and support each other's development. At the same time, study's results indicate that the learning environment created according to the KARİDE model and the proof problems in this environment enables proof to serve many purposes simultaneously, thus revealing the different functions of proof and supporting the development of each other. In teaching practices, it was observed that the verification function of proof was “Conviction” and “Confirmation,” the explanation function was “ Insight” and “Consequences,” the communication function was “Form of discourse,” and “Forum for debate,” the discovery function was “Analysis,” and the systematization function was “Inconsistencies,” “ Connections,” and “Application.” When the results are considered in the context of the relationship between norms and functions, it is evident that norms and proof functions intertwine within dialogues. Norms associated with discussion and collaboration themes mainly support the communication function of proof. In contrast, norms related to justification and problem-solving themes support the verification, explanation, discovery, and systematization functions of proof. The agreement within the class or small groups to share ideas, seek clarification on unclear points, defend their conjectures, and present opposing views facilitated student communication and created a classroom environment conducive to discussions. These results demonstrate that the norms within the discussion theme and the communication function of proof mutually support each other. At the same time, it was observed that the visibility of norms within the collaboration theme, which allowed small groups to produce solutions together, alongside the norms within the discussion theme, was made possible through the communication function of proof. The class's agreement on norms, such as explaining and justifying their views, providing acceptable justifications, avoiding experimental verification, offering different solutions to problems, and using algebraic methods in their solutions, indicated the presence of norms within the themes of justification and problem-solving. At the same time, these norms showed parallel progress with proof functions, such as verifying the proof by convincing oneself and the group, explaining why it is correct, discovering more general results, and revealing inconsistencies and connections between concepts. All these situations indicate that the mentioned norms and the functions of proof support each other. However, in this study, it was observed that the "Analysis" sub-function of the discovery function and the "Application" sub-function of the systematization function appeared less frequently in dialogues than in other functions. This result is due to the middle school level at which the study was conducted and the problems selected to suit this level.

The framework of mathematical proof addressed in this study is based on Stylianides' (2007) definition of proof, which is socially constructed and valid on classroom norms. It is known that the classroom community contributes to the formation of the microculture in the learning and teaching environment and is influenced by it. It is emphasized that understanding the role of a norm requires analyzing how it is related to other norms (Lopez & Allal, 2007). Indeed, the current study demonstrates how each norm is interconnected with different norms and how they mutually support one another in classroom dialogues. Furthermore, it is emphasized that social and socio-mathematical norms, when implemented alongside inquiry-based collaborative approaches in classrooms, are highly effective in ensuring students' active participation in mathematical discussions, enhancing the quality of their contributions, understanding a classroom's culture, and modifying this culture in the desired way (Partanen & Kaasila, 2015; Yackel & Cobb, 1996). In this classroom, designed according to the KARİDE model, where collaboration and mathematical discussions were actively conducted, it was observed that the norms under the discussion theme, in particular, enhanced the quality of the process of participating in classroom activities and solving proof problems. The students in this classroom agreed on sharing ideas, collaborating to produce solutions, asking questions about unclear points, presenting counter views, and forming a common solution guided and organized small group and whole-class discussions. Herbst et al. (2010) state that communication encompasses sharing the results reached in a proof, discussing them, and having their validity accepted or rejected by different individuals. Therefore, the results of this study demonstrate that the norms under the theme of discussion support the communication function of proof. As students shared their ideas while constructing proofs, the definitions, concepts, or justifications they used were evaluated by other students through the emerging forms of discourse. Thus, relationships were established between different justifications in the discussion environment created by the communication function of proof. While responding to counterarguments, students had the opportunity to revise their arguments, enabling them to identify inconsistencies between their arguments and those of their peers. Various researchers have emphasized that proof is significant in classroom settings for conveying arguments and demonstrating how students communicate with one another (Bartlo, 2013; Herbst et al., 2010). Cilli-Turner (2017) attributed the significant difference observed in the communication function of proof in a discussion-based classroom to the fact that traditional education hinders the communicative aspect of proof. Similarly, Bleiler-Baxter and Pair (2017), in their study investigating which classroom activities reveal the functions of proof, demonstrated that the most substantial relationship exists between the communication function of proof and discussion. Indeed, the consideration of proof as a social process in this classroom, facilitated by the KARİDE model, which enables small group and class discussions, supports the findings of these studies. In this process, students communicate using mathematical language, employ norms to ensure regularity in discussions and establish a framework for valid proofs.

In some proof problems, it has been observed that students engaged in diverse solution approaches and provided justifications during discussions about how the methods they used in their solutions differed from one another to convince group members and the class. Lev and Leikin (2017) state that students' use of diverse justifications and strategies encourages them to make discoveries. Indeed, in this study, the expectation for students to present different solutions led them to explore other approaches, provide varied justifications, generalize their solutions to apply to similar problems, and make discoveries. These activities facilitated the emergence of the discovery and systematization functions of proof. Similarly, Bartlo (2013) stated in his study that students can sometimes discover more general and more substantial conclusions by analyzing proof.

During discussions, students asking questions like, "Can you explain why this is correct?" or "How can you be sure it is true?" and the teacher initiating class discussions by asking, "Did this solution convince you?" demonstrates that both in small group and whole-class discussions, students reached a common understanding as they explained and justified their thoughts while trying to persuade their peers. At this point, it is evident that the norms within the theme of justification support both the explanation and verification functions of proof. For the explanations provided by students to convince group members or the class, they needed to include acceptable mathematical justifications or a valid mathematical proof. In this way, it has been highlighted by various researchers (Bartlo, 2013; Dreyfus, 1999; Herbst et al., 2010) that instead of passively accepting and memorizing mathematical explanations presented by others, students convince themselves of the validity of their mathematical ideas through proof can help them become autonomous learners. In parallel with the results of this study, Mudaly (2007), in his teaching experiment, focused on assisting students to understand that proof serves not only as a tool for verification but also for explanation. He demonstrated that students understood different functions of proof and acquired the skills to construct deductive proofs with proper justifications. The current study observed that during the initial weeks of the teaching experiment, the researcher-teacher communicated the expectation that students should provide explanations and justifications. In subsequent weeks, students became aware of this expectation without the teacher's repetition, leading to an insight into what constitutes an acceptable mathematical explanation and justification in this classroom. When an explanation or empirical verification did not align with this insight, it was observed that students responded to such explanations by presenting counterarguments, defending their viewpoints, refuting assumptions, and revealing inconsistencies through counterexamples. This situation demonstrated that the norms within the themes of justification and discussion are related not only to the explanation function of proof but also to its systematization function. Students asking about unclear points during discussions indicates the expectation for acceptable mathematical explanations. Dennis (2000) and De Villiers (1999) stated that the explanation function of proof makes visible how the concepts involved in the proof are related to one another and the given conjectures being proved and that the result obtained in the proof contributes to the systematization of knowledge by revealing the relationships within the proof. Similarly, Hanna (2000) stated that the explanation function of proof makes fundamental mathematical connections visible and demonstrates why a conjecture is true, leading to discoveries. In this classroom, students' ability to convince their peers of the correctness of their solutions depended on explaining their justifications, which needed to be acceptable mathematical justifications recognized by the class. In contrast, an acceptable mathematical justification and proof relied on establishing connections between various mathematical concepts. Yackel and Cobb (1996) stated that while constructing proofs, claims are justified, and the criteria for appropriate justifications are established simultaneously, with these criteria -defined as socio-mathematical norms- guiding classroom discussions. Ingram, Andrews, and Pitt (2019) state that justification is a communicative act, enabling the revelation of what students know and do not know about mathematical concepts through the justifications they present so that teachers can make more accurate pedagogical decisions through mathematical communication and justification. In this study, it was also observed that the norms within the theme of justification support the explanation and communication functions of proof.

The classroom microculture, shaped by the social and socio-mathematical norms co-constructed by the teacher and students, along with the teacher's pedagogical choices (including selected mathematical tasks, daily routines, expectations, and teaching strategies), is stated to influence students' proof-construction skills as well as their perspectives and understanding of proof (Martin & McCrone, 2003). The results of the current study align with other research in highlighting the teacher's role, strategies, and the significance of proof tasks selected by the teacher in establishing and developing norms within the classroom (Ozdemir-Baki & Kilicoglu, 2023; Sekiguchi, 2006). Ozgur (2017), in his study, stated that the mathematics teacher emphasized the importance of understanding why a claim is correct rather than merely showing that it is accurate, encouraging students to investigate, justify, and generalize their conjectures while supporting their understanding of proof by setting clear expectations about what constitutes an acceptable justification and proof. In this study, the teacher's clear expectations, established through norms in a discussion-based learning environment, enabled students to experience the different functions of proof. Indeed, the findings align with the recommendations of studies (Bleiler-Baxter & Pair, 2017; Rocha, 2019) that emphasize the need to meaningfully integrate all functions of proof into the learning environment, advocate for the inclusion of proof at the elementary level, and call for further research on how to structure proof experiences in these classrooms.

This study demonstrated how proof can be incorporated into middle school mathematics teaching by emphasizing its functions and social aspects through the teacher's pedagogical choices and practical guidance, supporting the necessity of making proof an integral part of mathematics education at all levels. However, the study highlighted certain elements that must be considered when incorporating proof into middle school mathematics teaching. In particular, it is necessary to select proof problems that employ forms of argumentation appropriate to the level of middle school students and conceptually accessible to them. At the same time, the teacher must present to students in advance the framework for the characteristics of a mathematically valid argument, the method of argumentation, and the forms of representation that will be accepted in the classroom. Before beginning implementations, as in the current study, the teacher should consider the existing classroom norms and create a discussion environment that facilitates the emergence of classroom norms supporting proof functions. In this context, integrating proof into the middle school mathematics curriculum as a tool for learning mathematics and designing classroom activities where students evaluate each other's reasoning through small group and class discussions using the KARİDE model, including practices aimed at establishing and developing social and socio-mathematical norms, can be recommended. Since the results of this study are limited to a single classroom and its microculture, it is recommended to investigate the functions of proof and the connections between these functions and norms at different levels and in diverse classroom settings. An increase in research in this direction could provide insights into the social aspect of proof and how it should be addressed at the middle school level.

References

Almeida, D. (2003). Engendering proof attitudes: Can the genesis of mathematical knowledge teach us anything? *International Journal of Mathematical Education in Science and Technology*, *34*(4), 479-488. doi:10.1080/0020739031000108574

Ayala-Altamirano, C., & Molina, M. (2021). Fourth-graders' justifications in early algebra tasks involving a functional relationship. *Educational Studies in Mathematics*, *107*(7), 359-382. doi:10.1007/s10649-021-10036-1

Baki, A. (2006). *Kuramdan uygulamaya matematik eğitimi*. Trabzon: Derya Kitabevi.

Bartlo, J. R. (2013). *Why ask why: An exploration of the role of proof in the mathematics classroom* (Doctoral dissertation). Portland State University, Portland, OR.

Battista, M. T. (2017). Mathematical reasoning and sense making. In M. T. Battista, J. M. Baek, K. Cramer, & M. Blanton, *Reasoning and sense making in the mathematics classroom: Grades 3-5* (pp. 1-26). Reston, VA: NCTM Store.

Bell, A. W. (1976). A study of pupils' proof-explanations in mathematical situations. *Educational Studies in Mathematics*, *7*(1), 23-40. doi:10.1007/BF00144356

Bleiler-Baxter, S. K., & Pair, J. D. (2017). Engaging students in roles of proof. *Education The Journal of Mathematical Behavior,* 47, 16-34. doi:10.1016/j.jmathb.2017.05.005

Campbell, T. G., Boyle, J. D., & King, S. (2020). Proof and argumentation in K-12 mathematics: A review of conceptions, content, and support. *International Journal of Mathematical Education in Science and Technology*, *51*(5), 754-774. doi:10.1080/0020739X.2019.1626503

Cervantes-Barraza, J. A., Moreno, A. H., & Rumsey, C. (2020). Promoting mathematical proof from collective argumentation in primary school. *School Science and Mathematics 120*(1), 4-14. doi:10.1111/ssm.12379

Cilli-Turner, E. (2017). Impacts of inquiry pedagogy on undergraduate students conceptions of the function of proof. *The Journal of Mathematical Behavior,* 48, 14-21. doi:10.1016/j.jmathb.2017.07.001

Chua, B. L. (2016). Justification in Singapore secondary mathematics. In P. C. Toh & B. Kaur (Eds.), *Developing 21st century competencies in the mathematics classroom* (pp. 165-188). World Scientific. doi:10.1142/9789813143623_0010

Cobb, P., Stephan, M., McClain, K., & Gravemeijer, K. (2001). Participating in classroom mathematical practices. *The Journal of the Learning Sciences*, *10*(1/2), 113-163. doi:10.1207/S15327809JLS10-1-2_6

Creswell, J. W. (2013). *Nitel araştırma yöntemleri: Beş yaklaşıma göre nitel araştırma ve araştırma deseni* (M. Bütün, & C. B. Demir, Ed. & Trans.). Ankara: Siyasal Kitabevi.

Çakır, A. (2021). *Üstün yetenekli öğrencilerin matematik sınıf kültürlerinin sosyo-matematiksel normlar bağlamında incelenmesi* (Unpublished doctoral dissertation). Marmara University, İstanbul.

Çakir, A., & Akkoç, H. (2024). Socio-mathematical norms related to problem solving in a gifted and talented mathematics classroom. *Mathematics Teaching Research Journal, 16*(1), 79-99.

Dennis, A. (2000). A survey of mathematics undergraduates' interaction with proof: some implications for mathematics education. *International Journal of Mathematical Education in Science and Technology, 31*(6), 869-890. doi:10.1080/00207390050203360

De Villiers, M. (1990). The role and function of proof in mathematics. *Pythagoras*, *24*(24), 17-24. Retrieved from https://www.researchgate.net/publication/26478464

De Villiers, M. (1999). *Rethinking proof with the geometer's sketchpad.* Oakland, CA: Key Curriculum Press.

Dogan, M. F. (2015). *The nature of middle school ın-service teachers' engagements in proving-related activities* (Doctoral dissertation). University of Wisconsin-Madison, USA.

Dreyfus, T. (1999). Why Johnny can't prove. *Educational Studies in Mathematics, 38*(1), 85-109. doi:10.1023/A:1003660018579

Fredriksdotter, H., Nor´en, N., & Bråting, K. (2022). Investigating grade-6 students' justifications during mathematical problem solving in small group interaction. *Journal of Mathematical Behavior, 67*(1), 1-19. doi:10.1016/j.jmathb.2022.100972

Hanna, G. (1983). *Rigorous proof in mathematics education*. Toronto, Ont.: OISE Press.

Hanna, G. (1995). Challenges to the importance of proof. *For the Learning of Mathematics*, *15*(3), 42-49. Retrieved from http://www.jstor.org/stable/40248188

Hanna, G. (2000). Proof, explanation and exploration: An overview. *Educational Studies in Mathematics,* 44, 5-23. doi:10.1023/A:1012737223465

Hanna, G., & Knipping, C. (2020). Proof in mathematics education, 1980-2020: An overview [Special issue]. *Journal of Educational Research in Mathematics*, 1-13. doi:10.29275/jerm.2020.08.sp.1.1

Harel, G., & Sowder, L. (2007). Toward a comprehensive perspective on on the learning and teaching of proof. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 1-60). Charlotte, NC: Information Age Pub. Inc.

Hemmi, K. (2010). Three styles characterizing mathematicians' pedagogical perspectives on proof. *Educational Studies in Mathematics*, *75*(3), 271-291. doi:10.1007/s10649-010-9256-3

Herbst, P., Miyakawa, T., & Chazan, D. (2010). *Revisiting the functions of proof in mathematics classrooms: A view from a theory of instructional exchanges*. Deep Blue at the University of Michigan. Retrived from http://hdl.handle.net/2027.42/78168

Hersh, R (2009). What I would like my students to already know about proof. In M. Blanton, D. Stylianou, & E. Knuth (Eds.), *Teaching and learning proof across the grades: K-16 perspective* (pp. 17-20). London: Routledge.

Ingram, J., Andrews, N., & Pitt, A. (2019). When students offer explanations without the teacher explicitly asking them to. *Educational Studies in Mathematics, 101*(2), 51-66. doi:10.1007/s10649-018-9873-9

Jahnke, H. N. (2010). The conjoint origin of proof and theoretical physics. In G., Hanna, H. N., Jahnke, & H., Pulte (Eds.), *Explanation and proof in mathematics: 17 philosophical and educational perpectives* (pp. 17-32). Berlin: Springer.

Jones, K., & Herbst, P. (2012). Proof, proving, and teacher-student interaction: Theories and contexts. In G. Hannai & M. De Villiers (Eds.), *ICMI study 19: Proof and proving in mathematics education* (pp. 261-279). Berlin: Springer.

Knuth, E. J. (2002). Teachers' conceptions of proof in the context of secondary school mathematics. *Journal of Mathematics Teacher Education*, *5*(1), 61-88. doi:10.1023/A:1013838713648

Levenson, E., Tirosh, D., & Tsamir, P. (2009). Students' perceived sociomathematical norms: The missing paradigm. *Journal of Mathematical Behavior, 28*(2-3), 171-187. doi:10.1016/j.jmathb.2009.09.001

Lev, M., & Leikin, R. (2017). The interplay between excellence in school mathematics and general giftedness: Focusing on mathematical creativity. In R. Leikin & B. Sriraman (Eds.), *Creativity and giftedness* (pp. 225-238). Berlin: Springer.

Lopez, M., & Allal, L. (2007). Sociomathematical norms and the regulation of problem solving in classroom microcultures. *International Journal of Educational Research, 46*(5), 252-265. doi:10.1016/j.ijer.2007.10.005

McClain, K., & Cobb, P. (2001). An analysis of development of sociomathematical norms in one first-grade classroom. *Journal for Research in Mathematics Education, 32*(3), 236-266. doi:10.2307/749827

Maher, C. A. (2009). Children's reasoning discovering the idea of mathematical proof. In M. Blanton, D. Stylianou, & E. Knuth (Eds.), *Teaching and learning proof across the grades: A K-16 curriculum* (pp. 120-132). London: Routledge.

Makar, K., & Fielding-Wells, J. (2018). Shifting more than the goal posts: Developing classroom norms of inquiry-based learning in mathematics. *Mathematics Education Research Journal, 30*(1), 53-63. doi:10.1007/s13394-017-0215-5

Martin, T. S., & Mccrone, S. (2003). Classroom factors related to geometric proof construction ability. *The Mathematics Educator*, *7*(1), 18-31. Retrieved from https://www.researchgate.net/publication/228574478

Martin, T. S., McCrone, S. M. S., Bower, M. L. W., & Dindyal, J. (2005). The interplay of teacher and student actions in the teaching and learning of geometric proof. *Educational Studies in Mathematics,* 60, 95-124. doi:10.1007/s10649-005-6698-0

Mason, J., Burton, L., & Stacey, K. (2010). *Thinking mathematically*. Pearson Education Limited.

Merriam, S. B. (2018). *Nitel araştırma desen ve uygulama için bir rehber* (S. Turan, Trans.). Ankara: Nobel Yayıncılık.

Meydan, A. (2021). Nitel araştırmalarda örnekleme yöntemleri. In A. Uzungöz (Ed.), *Bilimsel araştırma becerileri ve araştırmada güncel desenler* (pp. 47-61). Ankara: Pegem Akademi

Miles, M. B., & Huberman, A. M. (1994). *Qualitative data analysis: An expanded sourcebook*. Thousand Oaks, CA: SAGE.

Mudaly, V. (2007). Proof and proving in secondary school. *Pythagoras,* 66, 64-75. doi:10.4102/pythagoras.v0i66.81

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: NCTM.

Ozdemir Baki, G., & Kilicoglu, E. (2023). Social and socio-mathematical norms constructed by teachers in classes through the development of noticing skills. *International Electronic Journal of Mathematics Education, 18*(1), em0723. doi:10.29333/iejme/12649

Ozgur, Z. (2017). *Relationships between students' conceptions of proof and classroom factors* (Doctoral dissertation). University of Wisconsin-Madison, Madison.

Öksüz, H., & Gürefe, N. (2021). 5. Sınıf matematik öğretmenlerinin öğrenme güçlüğüne sahip öğrencilerin bulunduğu sınıfta oluşturmayı amaçladığı sosyomatematiksel normlar. *Cumhuriyet International Journal of Education, 10*(2), 601-626. doi:10.30703/cije.742571

Partanen, A. M., & Kaasila, R. (2015). Sociomathematical norms negotiated in the discussions of two small groups investigating calculus. *International Journal of Science and Mathematics Education*, *13*(4), 927-946. doi:10.1007/s10763-014-9521-5

Rocha, H. (2019). Mathematical proof: From mathematics to school mathematics. *Philosophical Transactions, 377*(2140), 2-12. doi:10.1098/rsta.2018.0045

Sekiguchi, Y. (2006). Mathematical norms in Japanese mathematics classrooms. In D. Clarke, C. Keitel, & Y. Shimizu (Eds.), *Mathematics classrooms in twelve countries: The ınsiders perspective* (pp. 289-306). Rotterdam: Sense.

Steffe, L.P. (1991). The constructivist teaching experiment: Illustrations and implications. In E. Von Glasersfeld (Ed.). *Radical constructivism in mathematics education. Mathematics education library* (pp. 177-194). Berlin: Springer. doi:10.1007/0-306-47201-5_9

Steffe, L. P., & Thompson, P. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds.), *Research design in mathematics and science education* (pp. 267- 307). Mahwah, NJ: Lawrence Erlbaum Associates.

Stylianides, A. J. (2007). Proof and proving in school mathematics. *Journal for Research in Mathematics Education, 38*(3), 289-321. doi:10.2307/30034869

Tall, D. (1998). The cognitive development of proof: Is mathematical proof for all or for some?. In Z. Usiskin (Ed.), *Developments in school mathematics education around the world* (pp. 117-136). Reston, VA: NCTM.

Tanışlı, D., & Yavuzsoy-Köse, N. (2013). Sınıf öğretmeni adaylarının genelleme sürecindeki bilişsel yapıları: Bir öğretim deneyi. *Elektronik Sosyal Bilimler Dergisi, 12*(44), 255-283. Retrieved from https://dergipark.org.tr/tr/download/article-file/70448

Toker, Z. (2020). Etkinlikler yoluyla sınıf içinde ispat ve sorgulama. In Y. Dede, M. F., Doğan, & F. Aslan-Tutak (Eds.), *Matematik eğitiminde etkinlikler ve uygulamaları* (pp. 439-463). Ankara: Pegem Akademi Yayınları.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education*, *27*(4), 458-477. doi:10.2307/749877

Yalman, E., & Uzungöz, A. (2021). Nitel araştırmalarda geçerlik ve güvenirlik. In A. Uzungöz (Ed.), *Bilimsel araştırma becerileri ve araştırmada güncel desenler* (pp. 125-140). Ankara: Pegem Akademi

Yıldırım, A., & Şimşek, H. (2006). *Sosyal bilimlerde nitel araştırma yöntemleri*. Ankara: Seçkin Yayıncılık.

Yılmaz, T. Y. (2021). *7. sınıf öğrencilerinin kanıtlama süreçlerinin ve bu süreçte ortaya çıkan kanıt işlevlerinin incelenmesi* (Unpublished doctoral dissertation). Anadolu University, Eskişehir.

Zaslavsky, O., Nickerson, S. D., Stylianides, A. Kidron, I., & Winicki-Landman, G. (2012). The need for proof and proving: mathematical and pedagogical perspectives. In G. Hanna & M. De Villiers (Eds.), *ICMI study 19: Proof and proving in mathematics education* (pp. 215-229). Berlin: Springer.

Appendix-1

Week 1 **Activity**

An integer is added to the integer two steps before it and the integer two steps after it. What can you say about this sum?

The Focus of the Activity

**Developing a mathematical conjecture*
**Determining whether the reached conjecture is always valid*
**Proving directly using algebraic representation (Direct Proof: Assuming the proposition p is true, the truth of proposition q is demonstrated using known definitions, theorems, and rules.)*
**Proving visually using a graphical representation*

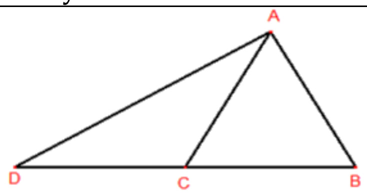
Week 2 **Activity**

For all integers n, $n^3 \geq n^2$.

The Focus of the Activity

**Examining the validity of the given proposition*
**Proving by counterexample (Providing a counterexample is a proof method where at least one example is given to show that the proposition is false.)*
**Updating the proposition as follows:*
If n is a natural number, then $n^3 \geq n^2$, If n is a negative integer, then $n^3 < n^2$

Week 3 **Activity**



$\overset{\Delta}{ADB}$ is a right triangle, $m(\widehat{DAC}) = 30^0$, $m(\widehat{CAB}) = m(\widehat{ABD}) = 60^0$ ve $|AB|$=6 cm. What can you say about the measure of angle D and the length $|DC|$?

The Focus of the Activity

**Developing mathematical conjectures*
**Determining whether the reached conjecture is always valid*
**Performing a direct proof.*

Week 4 **Activity**

Ali Baba is raising chickens and rabbits on his farm. The total number of heads of chickens and rabbits is 37, and the total number of legs is 98. How many chickens and rabbits does Ali Baba have on his farm?

Elaboration of the Activity:

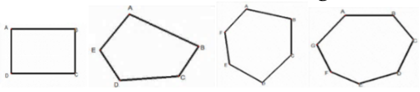
*If the total number of heads of chickens and rabbits on the farm were 40, and the total number of legs were 140, what would the result be?
*A group of 10 people, consisting of masters and workers, earns 128 lira from a job. If a master earns 20 lira and a worker earns 8 lira, how many workers and masters are in the group?

The Focus of the Activity

**Developing a general solution method*
**Applying the determined method to a similar problem*

Week 5 **Activity**

Below, a rectangle, a pentagon, a hexagon, and a heptagon are given. Find the sum of the measures of the interior angles of these geometric shapes.



Elaboration of the Activity:

*What is the sum of the measures of the interior angles of a 30-sided polygon?
*What is the sum of the measures of the interior angles of an n-sided polygon?
*How do we calculate the measure of one interior angle of a regular polygon?

The Focus of the Activity

**Developing a mathematical conjecture*
**Determining whether the reached conjecture is always valid*
**Demonstrating that the sum of the interior angles of a (convex) polygon is (n-2).180 by performing a direct proof*
**Establishing a geometry-algebra relationship: Providing a geometric explanation for the algebraic generalization*

Week 6 **Activity**

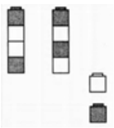
The unit digit of the square of an integer is always an element of the set A={0,1,4,5,6,9}.

The Focus of the Activity

**Examining the validity of the given proposition*
**Proving by exhaustion in a finite set (Exhaustive proof is a type of proof where all possible cases within the defined set of the proposition are tested one by one to demonstrate its truth.)*

Week 7 **Activity**

We have an unlimited number of black and white cubes. Using these cubes, we will build towers. Assuming at least one black cube and one white cube are used, how many different towers of height four can we construct, as shown?
If we were asked to build towers of height five, how many different towers could we construct?



The Focus of the Activity

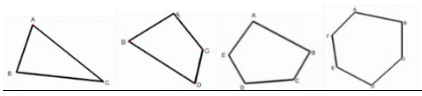
**Developing a mathematical conjecture*
**Determining whether the reached conjecture is always valid*
**Proving by exhaustion through systematic diagramming*

Week 8 **Activity**

What is the sum of the measures of the exterior angles of each polygon below?

Elaboration of the Activity:

*What is the sum of the measures of the exterior angles of a 30-sided polygon?
*What is the sum of the measures of the exterior angles of an n-sided polygon?



The Focus of the Activity

**Developing a mathematical conjecture*
**Determining whether the reached conjecture is always valid*
**Demonstrating through a direct proof that the sum of the measures of the exterior angles of a (convex) polygon is 360°*

Week 9 Activity

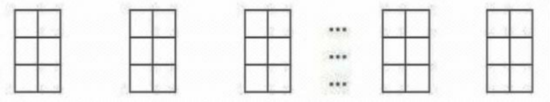
Every integer with a multiple of 6 is also a multiple of 3.

Ali: I tested some numbers multiples of 6, such as 12, 60, and 606, and observed that they are also multiples of 3. Therefore, this statement is always true.

Ayşe: Let n be an integer; any multiple of 6 can be written as 6n. We know that 6n = 3.2n, which is a multiple of 3. Therefore, any multiple of 6 is always a multiple of 3.

Ahmet: Suppose we have several cookies that is a multiple of 6. Let's distribute these cookies into several boxes, each containing 6 cookies. Then, we can further divide each box into 2 smaller boxes, containing 3 cookies. In this way, we have effectively distributed all the cookies into groups of 3. Therefore, any multiple of 6 is always a multiple of 3.

Aslı: The total number of square-shaped cards below is a multiple of 6.



Now I can arrange these square-shaped cards as shown below.

Therefore, any multiple of 6 is always a multiple of 3.

Order the arguments from the most convincing to the least convincing.

The Focus of the Activity

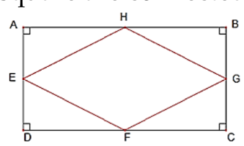
* *Evaluating the given arguments.*

Week 10 Activity

What can you say about the quadrilateral formed when the midpoints of the sides of a rectangle, as shown in the figure, are connected?

Elaboration of the Activity:

What can you say about the quadrilateral formed when the midpoints of the sides of a square are connected?



The Focus of the Activity

**Developing mathematical conjectures*

**Determining whether the reached conjecture is always valid*

**Demonstrating through direct proof that the quadrilateral (EFGH) is a rhombus.*

Week 11 Activity

The sum of five consecutive natural numbers is always a multiple of 5.

There are three consecutive integers whose sum is always a multiple of 6.

The sum of four consecutive natural numbers is always a multiple of 4.

The Focus of the Activity

**Examining the validity of the given propositions*

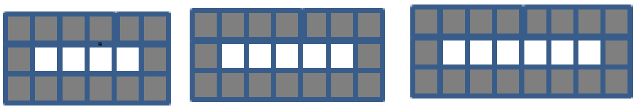
**In the first proposition, prove directly that the statement is always true.*

**In the second proposition, use the existence proof method to demonstrate that the statement is true (Existence proofs are a type of proof where giving a single example using existential quantifiers is sufficient to prove the statement).*

**In the third proposition, provide a counterexample to prove that the statement is false.*

Week 12 Activity

Ahmet is laying gray tiles in a single row around the white tiles, as shown in the figure. How many gray tiles are needed to surround (n) white tiles?



The Focus of the Activity

**Reaching a generalization*

**Deriving the general rule of the pattern and performing deductive verification by relating it to the figure.*